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LQ SERVO AND LQG/LTR CONTROLLER DESIGN FOR AN AIRCRAFT MODEL

In this paper we shall give a short view of the so-called Linear Quadratic Regulator theory, which can be consulted for more details in [1] and [2]. We are using Linear Quadratic Gaussian theory too, Kwakernaak and Silvan, Anderson and Moore, Davis and Vinter, Aström and Wittenmark, Franklin and Powel and many others worked on this theory. We revise the main stages for an LQ servo controller design, which was presented by Davison and Ferguson, Gangsaas, Kwakernaak and Silvan, Lewis and Stevens respectively.

The Linear Quadratic Gaussian/Loop Transfer Recovery method was elaborated by Doyle and Stein [12].

This article treats an example of an aircraft flight controller design, using mixed LQ servo and LQG/LTR methods.

We can refer to [3], [4] and [5], [7] where we can find some basic applications for Linear Quadratic controller design for simplified aircraft models. First of all we will examine the traditional optimal controller design of the aircraft flight controller system using LQ servo method. With LQG/LTR method we are recovering the stability margin of the feedback gain, based on LQ tracking method, at the plant input.

LQR AND LQ SERVO CONTROLLER DESIGN

In this chapter we revise the basic of the LQ tracker controller method. The state-dynamic and measurement equation of the system is given by:

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx$$

$$u = -Kx + r \tag{2}$$

where x is the state of the system, u is the control input, y is the output, r is the reference command signal. The A, B, C matrices are the parameter matrices of the system, and K is the feedback gain. The equation (2) shows how we can provide the control input. We suppose the effect of control input does not appear in the measurement equation. We are able to receive the required Linear Quadratic Regulator if we minimize the functional.

$$J = \int_0^T (x^T Q x + u^T R u)dt \rightarrow \min_u \tag{3}$$

where *Q* and *R* are the weighting matrices. We shall design such a controller which could follow a predict signal, a nonzero reference command signal. If we examine the pure Linear Quadratic Regulator we can conclude that we always have tracking error. The key to achieve a required performance is in entering an integrator to the system. In practice we augment the state number of the system, we add some supplement states to the original system. Firstly, we separate the states in two groups. One group will contain the states that act directly to output, the other part groups the remained states. This separation is needed to determine the number of supplement integrator to include in the system.

$$\dot{x} = Ax + Bu, \quad y_P = C_p x, \quad x_r = D_p x \tag{4}$$

With C_p we select those states, which are outputs of the system. With D_p we select those states, which are the remained states of the system. After separating the states in this form we augment the number of states with z_p. So the plant becomes:

$$\hat{x} = \begin{bmatrix} y_p \\ x_r \\ z_p \end{bmatrix}, \qquad \begin{bmatrix} \dot{y}_p \\ \dot{x}_r \\ \dot{z}_p \end{bmatrix} = \left[\begin{array}{c|c} A & 0 \\ \hline -C_p & 0 \end{array}\right] \begin{bmatrix} y_p \\ x_r \\ z_p \end{bmatrix} + \begin{bmatrix} B \\ \hline 0 \end{bmatrix} u \tag{5}$$

We will also design the K feedback gain for the augmented plant, with LQR method. The LQ servo plant can be seen in the following *Figure 1*:

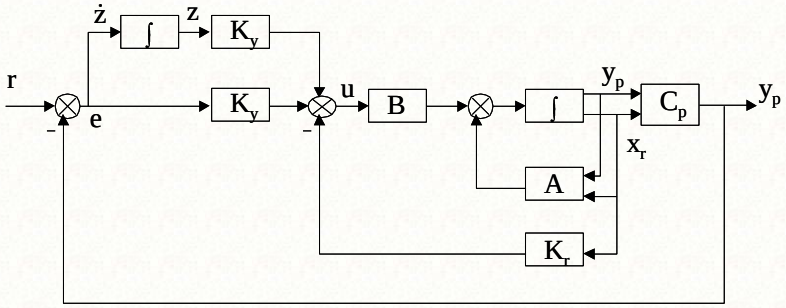


Fig. 1. Block structure of the LQ tracker problem

In this case the control input can be written:

$$u = K_y(r - y) + K_z z - K_r x_r \tag{6}$$

The open-loop system in state-space form is given by:

$$\begin{aligned}\dot{x} &= Ax + Bu = Ax + B(K_y e + K_z z - K_r x_r)\\ \dot{x} &= (A - BK_r D_p)x + BK_z z + BK_y e\\ y_p &= C_p x\\ \dot{z} &= e\end{aligned} \tag{7}$$

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}A - BK_r D_p & BK_z\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}BK_y\\ I\end{bmatrix} e \tag{8}$$

$$y_p = \begin{bmatrix}C_p & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}$$

The closed-loop system in state-space form is given by:

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}A - BK_r C_p - BK_r D_p & BK_z\\ -C_p & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}BK_y\\ I\end{bmatrix} r \tag{9}$$

$$y_p = \begin{bmatrix}C_p & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix}$$

We suppose in this case that all states are avaible to measure. In general this is not the reality, so we must design a state observer (for example Kalman filter) to reconstruct the states.

GENERAL LQG/LTR PROBLEM

As we know well, both the LQ regulator, and the Kalman filter have good robustness and performance, so the LQG controller would have good properties too. Unfortunately this is not the case. We have two methods for the LQG design. Either we design the feedback gain before the Kalman filter, or we make first a filter and a feedback gain after. We receive different solution if we use the two different methods of the LQG design. Are we able to find a way for tuning the LQG controller with a parameter? There is a way of recovering either the full state-feedback properties or the state estimator properties. Let introduce the following notations, *x* is the state vector, *r* is the referential signal, *y* is the output vector, *w(t)* is the state noise, *v(t)* is the sensor noise, $\hat{x}$ is the estimated state vector, e is the error signal, $\hat{y}$ is the estimated output vector, *A,B,C* are the state parameter matrix, *G* is the disturbance input matrix, *K* is the static feedback gain matrix, *L* is the stationary Kalman filter matrix, the observer gain, ε is the state error and *I* is the identity matrix. Let the state space representation of the plant and its transfer function is the following:

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw \\ y &= Cx + v \\ G(s) &= C(sI - A)^{-1}B \end{aligned} \tag{10}$$

G(s) is the Laplace transform of the transfer function from control input to output. The controller is given by:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - L(r + \hat{y}) \\ \hat{y} &= C\hat{x} \\ u_{opt} &= -K\hat{x} \\ \dot{\hat{x}} &= (A - BK - LC)\hat{x} - Le \\ G_C(s) &= K(sI - A + BK + LC)^{-1}L \end{aligned} \tag{11}$$

$G_C(s)$ is the Laplace transform of the transfer function of the controller.

The closed–loop can be written as:

$$\begin{aligned} \varepsilon &= x - \hat{x} \\ \dot{x} &= Ax + Bu = Ax - BKx + BK\varepsilon \\ \dot{\varepsilon} &= Ax + Bu - A\hat{x} - Bu + L(r - C\hat{x}) = (A - LC)\varepsilon + Lr \\ \begin{bmatrix} \dot{x} \\ \dot{\varepsilon} \end{bmatrix} &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \end{bmatrix} + \begin{bmatrix} 0 \\ L \end{bmatrix} r \\ y &= \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \varepsilon \end{bmatrix} \end{aligned} \tag{12}$$

The transfer function of the open – loop:

— for input $G^{I}_{H}(s) = G_C(s)G(s)$ (13)

— for output $G^{O}_{H}(s) = G(s)G_C(s)$ (14)

LQG/LTR PROBLEM AT THE PLANT INPUT

First, we will write the open loop transfer function

$$G_H = K(sI - A + BK + LC)^{-1}LC(sI - A)^{-1}B \tag{15}$$

Applying the following substitutions:

$$\Phi(s) = (sI - A)^{-1}$$
$$\Psi(s) = (sI - A + BK)^{-1} \quad (16)$$

The open loop transfer become more simple:

$$G_H(s) = K(\Psi(s)^{-1} + LC)^{-1} LC\Phi(s)B \quad (17)$$

We utilize now the following matrix lemma:

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1} \quad (18)$$

$$A = \Psi(s)^{-1} \qquad B = L \qquad C = I^{-1} \qquad D = C$$

$$G_H(s) = K(\Psi(s) - \Psi(s)L(C\Psi(s)L + I))^{-1} C\Psi(s))LC\Phi(s)B =$$
$$= K\Psi(s)(I - (LC\Psi(s) + I)^{-1} LC\Psi(s))LC\Phi(s)B = \quad (19)$$
$$K\Psi(s)(LC\Psi(s) + I)^{-1} LC\Phi(s)B$$

The calculation of the observer, the Filter Algebric Ricatti Equation (FARE)

$$L = PC^T V^{-1}$$
$$AP + PA^T + \Gamma W \Gamma^T - PC^T V^{-1} CP = 0 \quad (20)$$
$$W = W_0 + qM$$

where P is the solution of the FARE. We will examine the situation while $q \to \infty$:

$$AP + PA^T + \Gamma W_0 \Gamma^T + q\Gamma M \Gamma^T - PC^T V^{-1} CP = 0$$
$$\frac{AP}{q} + \frac{PA^T}{q} + \Gamma M \Gamma^T - \frac{PC^T V^{-1} CP}{q} = 0 \quad (21)$$

$$\lim_{q \to \infty} \frac{P}{q} = 0 \quad (22)$$

So when $q \to \infty$ L has got a solution:

$$\Gamma M\Gamma^T = \frac{PC^T V^{-1} CP}{q}$$

$$q^{1/2}\Gamma M^{1/2} = PC^T V^{-1/2} \tag{23}$$

$$\lim_{q\to\infty} L = q^{1/2}\Gamma M^{1/2} V^{-1/2}$$

Shall we choose $\Gamma = B \quad M = I$! The open loop transfer function

$$\lim_{q\to\infty} L = q^{1/2} B V^{-1/2} \tag{24}$$

$$\lim_{q\to\infty} G_H(s) = K\Psi(s)(q^{1/2} B V^{-1/2} C\Psi(s) + I)^{-1} q^{1/2} B V^{-1/2} C\Phi(s)B =$$

$$= K\Psi(s)(BV^{-1/2} C\Psi(s) + q^{-1/2})^{-1} BV^{-1/2} C\Phi(s)B =$$

$$= K\Psi(s)\Psi(s)^{-1} C^{-1} V^{1/2} B^{-1} BV^{-1/2} C\Phi(s)B =$$

$$\lim_{q\to\infty} G_H(s) = K\Phi(s)B \tag{25}$$

The stages of the design LQG/LTR for input

— We must find the required LQ loop gain, K.

— Afterwards we determine the Kalman filter, using the substitutions: $\Gamma = B \quad M = I \quad W = W_0 + qI \quad V = I$. We search the L observer gain while $q \to \infty$.

We must take care of this type of LQ tuning, because we enhance the state noises while we approach the required LQ gain. The control input is limited!

In case a non-minimal phase system the optimal loop cannot be perfectly rebuilt because the zeros of the open loop transfer function are not all positives.

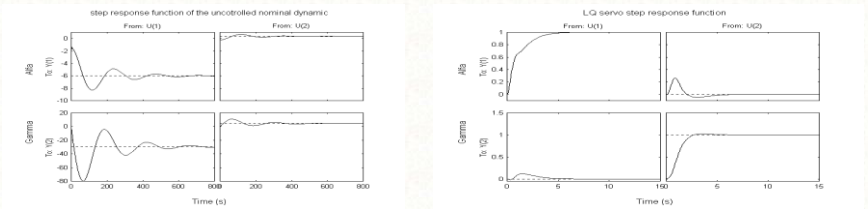
MIXED LQ SERVO AND LQG/LTR CONTROLLER DESIGN FOR AN AIRCRAFT MODEL

During the design of the mixed LQ servo and LQG/LTR controller design it is supposed, that our aircraft is cruising at constant altitude 30 000 ft, and with constant velocity 0.75 M. The linearized longitudinal equations are simple, ordinary linear differential equations with constant coefficients. The coefficients in the differential equations are made up of aerodynamic stability derivatives, mass and inertia characteristics of the aircraft. We can write these equations in state-space form:

$$\begin{bmatrix}\dot{q}\\ \dot{u}\\ \dot{\alpha}\\ \dot{\theta}\end{bmatrix}=\begin{bmatrix}-0.8 & -0.0006 & -13.2 & 0\\ 0 & -0.014 & -16.64 & -32.2\\ 1 & -0.0001 & -1.65 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}q\\ u\\ \alpha\\ \theta\end{bmatrix}+\begin{bmatrix}-19 & -2.5\\ -0.66 & -0.5\\ -0.16 & -0.6\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_E\\ \delta_f\end{bmatrix} \quad (26)$$

$$\begin{bmatrix}\theta\\ \gamma\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}q\\ u\\ \alpha\\ \theta\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_E\\ \delta_f\end{bmatrix}$$

where control inputs are $\delta_f(t)$ is the perturbed flapperon angle deflection, $\delta_E(t)$ is the perturbed elevator angle deflection, where states are $\alpha(t)$ is the perturbed angle of attack, $\theta(t)$ is the perturbed pitch angle, q(t) is the pitch rate, u(t) is the perturbed horizontally velocity and γ is the flight path angle (θ - α). We are controlling the pitch angle with elevator deflection, and flight path angle with flaps. Before we design compensator for our hypothetical aircraft dynamics it is expected to analysis the pure dynamics. We study the transient response to specific test signals. The step function and Dirac's delta or impulse function are chosen as testing inputs. We can conclude some information about the transient behaviour. The uncontrolled dynamic is stable.



a) Uncontrolled dynamic b) LQ servo controlled plant

Fig. 2. Step response functions

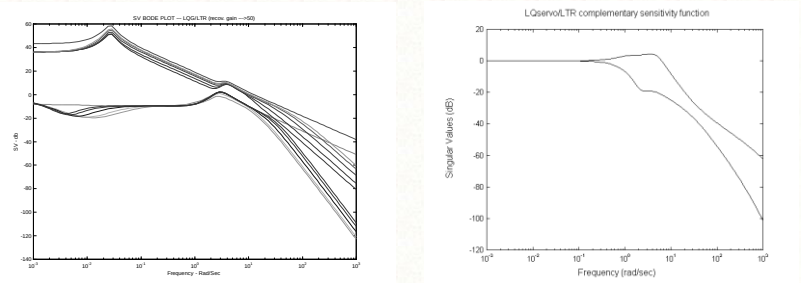
The short and long period (phugoid) oscillation of the uncontrolled aircraft dynamics is two special characteristics of the aircraft movement. They are caused by the special placement of the poles. In this case we have p1=-1.1512 + 3.4464i, p2= -1.1512 - 3.4464i, p3= -0.0058 + 0.0264i, p4=-0.0058 - 0.0264i. The poles, which are close to the imaginary axe, simulate the long period movement, and the other pole-pair conjugated cause the short period dynamic. The periods are readily obtained once the eigenvalues are known. Our linear system is controllable and observable, and because of the equality of the Kalman rank of the observability and controllability matrix it is minimal too. We augment the state number of the plant with 2. Using the Inverse Square Rule we are able to determine the weighting matrices *Q* and *R* for the augmented plant as follows:

$$Q = \begin{bmatrix} 0.0365 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.000025 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.672 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

In the *Figure 2.b* we can see the step response function of the controlled plant, with LQ tracker controller. As we know well, in our case we could not measure all states. This is the reason of design an state observer. Applying the separation principle, we designed the K_{SRV} feedback gain, and now, for the augmented plant we compute the L static observer gain of the Kalman filter. Let the covariance matrices of the state and sensor noises be W and V.

$$W = \begin{bmatrix} 0.001 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.001 \end{bmatrix}, \quad V = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$$

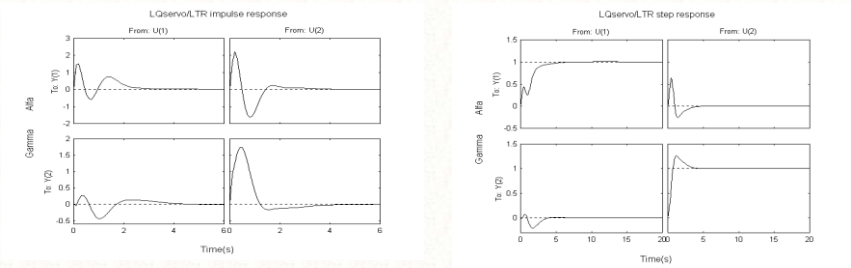
If we accept the stability margin of the LQ servo controlled plant, increasing q ($q_{final}=50$), we are able to recover it. When we design the loop transfer recovery at the plant input method we want to get back, in a frequency-band relatively large, the properties of the open loop transfer function of the LQ servo controlled plant.



a) Open-loop singular value diagram b)Closed-loop singular value diagram

Fig. 3. Frequency plot of the mixed LQ servo and LQG/LTR controlled plant

In the *Figure 3* and *4* we can see the frequency and time analysis of the robust servo controller, using LQ tracker and LQG/LTR method. We conclude the servo properties of the system.

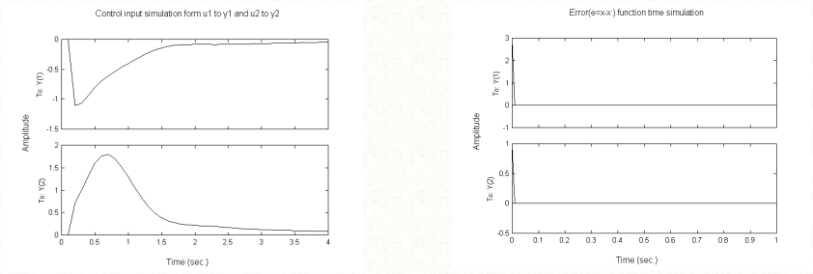


a) Closed-loop simulation, impulse response b) Closed-loop simulation, step response

Fig. 4. Time simulation of the mixed LQ servo and LQG/LTR controlled plant

SUMMARY

This article treats an example of an aircraft flight controller design, using mixed LQ servo and LQG/LTR methods, and shows to advantage the robust servo controller. We had success to recover the K_{srv} feedback gain, which permits the nonzero reference signal tracker property. The recover is not totally fulfilled because, prima, our system is non-minimal phase, and than, the recover is only an approximation. The *Figure 5* illustrates the control input signal simulation, and the error function simulation of the stochastic state observer.



a) Control input simulation b) Error simulation of the Kalman filter

Fig. 5. Time simulation of the mixed LQ servo and LQG/LTR controlled plant

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