# ROBUST CONTROLLER SYNTHESIS FOR THE AIRCRAFT PITCH ATTITUDE CONTROL SYSTEM

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## INTRODUCTION

The most important requirement, which is prescribed for the automatic flight control systems (AFCSs) is the quick tracking ability with respect to a reference signal. The other important design goal is the simultaneous capability as much as possible to reject the unwanted effects from the internal and external disturbances [4,7]. These two requirements conflict and the control system design may be achieved as a consequence of some compromise. The purpose of the author is to summarize main equations of the LQR and LQG design methods and to present a numerical example for application of design methods mentioned above.

## TRADITIONAL OPTIMAL CONTROL LAW SYNTHESIS BASED UPON LQR DESIGN METHOD

The most common design performance criterion of the modern optimal control theory used in the aeronautical sciences is the integral performance index given by [1,2,3] to be:

$$J = \frac{1}{2}\int_0^\infty \left( \mathbf{x}^\mathrm{T}\mathbf{Q}\,\mathbf{x} + \mathbf{u}^\mathrm{T}\mathbf{R}\,\mathbf{u} \right) dt \to Min \qquad (1)$$

where **x** and **u** are state and the input vectors, **Q** is a positive definite, **R** is a positive definite or positive semidefinite weighting matrices, respectively.

The problem to be solved is as follows: for a given aircraft with the linear time invariant (LTI) model find the control vector **u**, which will minimize the performance criterion (1). This problem also called as the minimum energy control problem. The linear optimal control law, which is minimizing the performance index (1) defined by [1,2,3] as follows :

$$\mathbf{u^o} = -\mathbf{K}\,\mathbf{x} \tag{2}$$

where **K** is the feedback gain matrix.

It is asummed, that the dymanics of the aircraft is given with its state space representation in the body–fixed coordinate system in the following manner [1,2,3]:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}\ \ , \mathbf{y} = \mathbf{C}\,\mathbf{x} + \mathbf{D}\,\mathbf{u}\,, \tag{3}$$

where **y** is the output vector, **C** and the **D** are the output and the feedforward matrices, respectively.

Substituting the optimal control law (2) into the 1st equation of (3) results in:

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} - \mathbf{B}\,\mathbf{K}\,\mathbf{x} = \left(\,\mathbf{A} - \mathbf{B}\,\mathbf{K}\,\right)\mathbf{x} \tag{4}$$

Supposing that the eigenvalues of the matrix $\left[\mathbf{A} - \mathbf{B}\,\mathbf{K}\right]$ have negative values or, if there is any complex with negative real part. Substituting eq. (4) into (1) leads to the following cost function:

$$J = \frac{1}{2}\int_0^\infty \left(\,\mathbf{x^T Q x} + \mathbf{x^T K^T R K x}\,\right)dt = \frac{1}{2}\int_0^\infty \mathbf{x^T}\left(\,\mathbf{Q} + \mathbf{K^T R K}\,\right)\mathbf{x}\,dt \to Min \tag{5}$$

The minimization of the quadratic integral criterion (5) can be achieved using the second method of Liapounov. The second method of Liapounov states, that for any **x** state vector takes place the next equation

$$\mathbf{x^T}\left(\,\mathbf{Q} + \mathbf{K^T R K}\,\right)\mathbf{x} = -\frac{d}{dt}\left(\,\mathbf{x^T P x}\,\right) \tag{6}$$

where **P** is a positive definite or real symmetric matrix (cost matrix). Taking derivative from the right side of eq (6) results in

$$\mathbf{x^T}\left(\mathbf{Q} + \mathbf{K^T R K}\right)\mathbf{x} = -\dot{\mathbf{x}}^{\mathbf{T}}\mathbf{P x} - \mathbf{x^T P}\dot{\mathbf{x}} = -\mathbf{x^T}\left[\left(\mathbf{A} - \mathbf{B K}\right)^{\mathbf{T}}\mathbf{P} + \mathbf{P}\left(\mathbf{A} - \mathbf{B K}\right)\right]\mathbf{x} \tag{7}$$

By the means of the second method of Liapunov for a given positive definite matrix $\left[\mathbf{Q} + \mathbf{K^T R K}\right]$ there is exists a positive definite matrix **P** such that

$$(\mathbf{A}-\mathbf{BK})^{\mathbf{T}}\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{BK})=-(\mathbf{Q}+\mathbf{K}^{\mathbf{T}}\mathbf{RK}) \tag{8}$$

In this case the integral performance criterion $J$ can be rewritten in the following manner:

$$\mathbf{J}=\frac{\mathbf{1}}{\mathbf{2}}\int_{\mathbf{0}}^{\infty}\mathbf{x}^{\mathbf{T}}\left(\mathbf{Q}+\mathbf{K}^{\mathbf{T}}\mathbf{RK}\right)\mathbf{x}\mathbf{dt}=-\left[\mathbf{x}^{\mathbf{T}}\mathbf{Px}\right]_{\mathbf{0}}^{\infty}=-\mathbf{x}^{\mathbf{T}}(\infty)\mathbf{Px}(\infty)+\mathbf{x}^{\mathbf{T}}(0)\mathbf{Px}(0) \tag{9}$$

It was assummed, that all eigenvalues of the matrix $[\mathbf{A}-\mathbf{B}\,\mathbf{K}]$ have negative real parts, we have in this case $\mathbf{x}\ (\infty) \to 0$. Therefore, the quadratic integral criterion may be written as follows

$$J=\mathbf{x}^{\mathbf{T}}(0)\mathbf{Px}(0) \tag{10}$$

The weighting matrix $\mathbf{R}$ is positive definite Hermitian or real symmetric matrix. One can write that:

$$\mathbf{R}=\mathbf{T}^{\mathbf{T}}\,\mathbf{T} \tag{11}$$

where $\mathbf{T}$ is the nonsingular matrix. Then eq (8) can be rewritten in the following manner:

$$\left(\mathbf{A}^{\mathbf{T}}-\mathbf{K}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\right)\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{BK})+\mathbf{Q}+\mathbf{K}^{\mathbf{T}}\mathbf{T}^{\mathbf{T}}\mathbf{TK}=0\text{, or} \tag{12}$$

$$\mathbf{A}^{\mathbf{T}}\mathbf{P}+\mathbf{PA}+\left[\mathbf{TK}-\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]^{\mathbf{T}}\left[\mathbf{TK}-\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]-\mathbf{PBR}^{-1}\,\mathbf{B}^{\mathbf{T}}\mathbf{P}+\mathbf{Q}=0 \tag{13}$$

The minimization of $J$ with respect to $\mathbf{K}$ requires the minimization of

$$\left[\mathbf{TK}-\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right]^{\mathbf{T}}\left[\mathbf{TK}-\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}\right] \tag{14}$$

Since eq (14) is nonnegative, its minimum occurs, when it is zero or, in that case when

$$\mathbf{TK}=\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P} \tag{15}$$

Hence the optimal feedback gain matrix can be found as

$$\mathbf{K}^{\mathbf{0}}=\mathbf{T}^{-1}\left(\mathbf{T}^{\mathbf{T}}\right)^{-1}\mathbf{B}^{\mathbf{T}}\mathbf{P}=\mathbf{R}^{-1}\,\mathbf{B}^{\mathbf{T}}\mathbf{P} \tag{16}$$

Equation (16) determining the feeback gain matrix of the optimal control law defined by eq (2). In case when eq (15) takes place eq (13) can be rewritten as follows

$$\mathbf{A}^{\mathbf{T}}\mathbf{P}+\mathbf{PA}-\mathbf{PBR}^{-1}\,\mathbf{B}^{\mathbf{T}}\mathbf{P}+\mathbf{Q}=\mathbf{0} \tag{17}$$

Equation (17) also called as the reduced - matrix Ricatti equation(algebraic Ricatti equation – ARE) for the time invariant **P** matrix. The optimal control system design contents the following two steps:

- Solution of the ARE — eq (17) — for determination of the cost matrix **P**.
- Substituting matrix **P** into eq (16). The resulting feedback gain matrix **K** is optimal for the chosen **Q** and **R** matrices.

# TRADITIONAL OPTIMAL DESIGN OF THE AFCS USING LQG DESIGN METHOD

The stochastic system optimization is often called as Linear Quadratic Gaussian (LQG) problem [1,3]. The LGQ control design is achieved in two stages using the separation principle. First design phase is the solution of the LQR problem (LQR – Linear Quadratic Regulator). The second stage is solution of the LQE problem (LQE – Linear Quadratic Estimator), in other words, the optimal Kalman-Bucy filter design problem.

## DYNAMIC MODEL OF THE FIGHTER AIRCRAFT

The longitudinal motion dynamic model of the aircraft is in [4,7] as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\ ,\ \mathbf{y} = \mathbf{C}\mathbf{x} \tag{18}$$

where: $\mathbf{x}^{\mathbf{T}} = [\mathrm{v}_x \quad \Theta \quad \omega_z \quad \alpha]$ – the state vector transpose with the following state variables : $v_x$ – speed component along longitudinal axis of the stability axis system; $\Theta = \vartheta - \alpha$ – flight path angle; $\omega_z$ – pitch rate; $\alpha$ – angle-of-attack; $\vartheta$ -pitch angle; $\mathbf{u}^{\mathbf{T}} = [\delta_{TH} \quad \delta_E]$ – input vector transpose with the input variables: $\delta_{TH}$ – change in thrust $\delta_E$ – elevator angular deflection; **y** – output vector; **A** – state matrix ; **B** – input matrix , **C** – output matrix.

The hypothetical aircraft data have been taken from [7] to be:

$$\mathbf{A}=\begin{bmatrix} -0.0204 & -0.0208 & 0 & -0.0466 \\ 0.163 & 0 & 0 & 2 \\ -1.021 & 0 & -1.8 & -2638 \\ -0.163 & 0 & 1 & -2 \end{bmatrix};\ \mathbf{B}=\begin{bmatrix} 0.027 & 0 \\ 0 & 0 \\ 0 & 60 \\ 0 & 0 \end{bmatrix};\ \mathbf{C}=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

During design of the LQG controller it is supposed, that aircraft is cruising at constant heigth of the flight with constant speed. The aircraft dynamics is supposed to be represented only with mathematical model of the short period motion. The following Euler kinematic equation is considered between the pitch rate and the pitch angle [4,7]:

$$\dot{\vartheta} = \omega_z \cos\gamma + \omega_y \sin\gamma \tag{20}$$

It is supposed that the lateral motion parameters such as $\omega_y, \gamma$ have negligible small values.

# THE LQG PROBLEM APPLIED FOR THE OPTIMIZATION OF THE AIRCRAFT PITCH ATTITUDE CONTROL SYSTEM

When the aircraft motion is analysed in the real environment, its dynamic model is corrupted by the external load factor — e.g. air turbulence — and by the measurement noises. The aircraft linear model representing its longitudinal motion, when the external stochastic turbulent air and internal random measurement noises are considered in the mathematical model can be defined as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{\Gamma}\mathbf{w}, \; \mathbf{y} = \mathbf{C}\mathbf{x} + \upsilon \tag{21}$$

where: $\mathbf{w}$ and $\upsilon$ are random external disturbance input vector and sensor noise vector, respectively. Both are white Gaussian zero-mean stationary processes with known covariances, $\mathbf{\Gamma}$ is disturbance input matrix.

The problem to be solved is as follows: find a stabilizing LQG controller, which will minimize the following average integral performance index (cost function) [1,3]:

$$J = \lim_{T\to\infty} \frac{1}{2T} E\left\{ \int_0^T \left( \mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\,\mathbf{u} \right) dt \right\} \to Min \tag{22}$$

There is the separation principle states that the problem may be solved in two separate stages.

1) LQR design stage. The solution is found by solving the following reduced-matrix Ricatti equation:

$$\mathbf{u} = -\mathbf{K}\,\mathbf{x(t)}; \mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}; \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} = 0 \tag{23}$$

where : **K** is the static feedback gain matrix, **P** is a positive definite cost matrix.

2) LQE design stage. The solution is found by solving the filter Ricatti equation given below [2,3,7]:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}); \mathbf{L} = \boldsymbol{\Sigma}\mathbf{C}^T\mathbf{R}^{-1}; \mathbf{A}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{A}^T - \boldsymbol{\Sigma}\mathbf{C}^T\mathbf{R}_o^{-1}\mathbf{C}\boldsymbol{\Sigma} + \boldsymbol{\Gamma}\mathbf{Q}_o\boldsymbol{\Gamma}^T = 0 \quad (24)$$

where $\mathbf{L}$ is a stationary Kalman filter static gain, $\boldsymbol{\Sigma}$ is a positive definite matrix, $\hat{\mathbf{x}}$ estimate of the the state vector and $\mathbf{R}_o$ , $\mathbf{Q}_o$ are weighting matrices of the state and the input vectors used during solution of the LQE problem.The simplified block diagram of control system of the aircraft longitudinal motion can be seen in Figure 1.

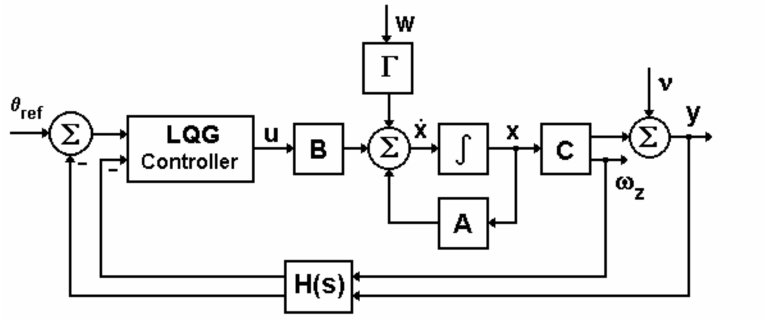


Figure 1. Simplified Block Diagram of the Longitudinal Motion

## SELECTION OF WEIGHTING MATRICES

The main goal of the application of the LQG design method is to find the stabilizing controller for the aircraft, which provides robustness for stabilization of the pitch angle in the turbulent air through minimizing of the sensitivity of the control system with respect to the external disturbances.

Let us find elements for weighting matrices in order to have response of the closed loop flight control system with dynamic performance of $\xi = 0{,}7$ [6].

Let us find weigths for the LQE design: the weighting matrices were chosen a constant during optimization as they are listed below

$$\mathbf{Q}_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \mathbf{R}_o = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (25)$$

Le us find weigths for the LQR design: the first set of weights is achieved using the so-called inverse square rule explained in [5,7]. Supposing that all state variables of the cost function are intended to be equally important, the elements of the weighting matrices can be found using the method of determination of the maximum values of the state variables and use these parameters while we find the elements of weights. For the single-engined delta-wing fighter aircraft the limitations in the pitch attitude flight control system are as follows:

$$|\delta_E| \le 1\,deg; |\omega_z| \le 18\,deg/sec; |\vartheta| \le 80\,deg \quad (26)$$

Setting each term to unity in the cost function of the LQR — when all state variables are at theirs limits — results in the following weighting matrices:

$$\mathbf{Q_1} = \begin{bmatrix} 3{,}086*10^{-3} & 0 \\ 0 & 1{,}562*10^{-4} \end{bmatrix};\ r_1 = 1 \qquad (27)$$

# COMPUTER AIDED DESIGN OF THE PITCH ATTITUDE CONTROL SYSTEM

When the computer aided design of the control system has been made, the damping ratio of the closed loop system was found. The damping ratio was $\xi_1 = 0.7795$, which is bigger its desired value of 0.7.

For weighting matrices (25) and (27) the Kalman filter static gain and the state feedback gain was found to be:

$$\mathbf{L} = \begin{bmatrix} 0.0822 & 1.0759 \\ 0.2575 & 0.0822 \end{bmatrix}; \mathbf{K_1} = \begin{bmatrix} 0.5556 & 0.1200 \end{bmatrix} \qquad (28)$$

At second attempt elements of the LQR weighting matrices has been varied and heuristically set as follows

$$\mathbf{Q}_2 = \begin{bmatrix} 3 & 0 \\ 0 & 0.3 \end{bmatrix}; r_2 = 1 \qquad (29)$$

For weighting matrices (29) the static feedback gain matrix **K** was found to be:

$$\mathbf{K}_2 = \begin{bmatrix} 1.7321 & 0.5689 \end{bmatrix} \qquad (30)$$

Checking properties of the closed loop control system the damping ratio of the transient behaviour is as follows: $\xi_2 = 1.0000$, which is more than it was in the previous attempt.

At attempt elements of the LQR weighting matrices has been varied and heuristically set as follows:

$$\mathbf{Q}_3 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.00003 \end{bmatrix} \qquad (31)$$

After setting new weighting matrices the optimal static feedback gain matrix **K** of the control system is as follows:

$$\mathbf{K}_3 = \begin{bmatrix} 0.5477 & 0.1085 \end{bmatrix} \tag{32}$$

The closed loop control system properties where checked and the damping ratio was found: $\xi_3 = 0.7248$ .

For these three typical sets of weighting matrices the time domain behavior of the closed loop pitch attitude control system has been analized when the input is a reference value of the pitch angle. The transient responses of the inner loop (stability augmentation system) and the outer loop (unity feedback by the pitch angle ) can be seen in Figure 2 and in Figure 3, respectively.

## RESULTS OF THE COMPUTER SIMULATION

During the computer simulation the input was the reference value of the pitch angle of the aircraft and the output variables were pitch angle and the pitch rate. In Figure 2 can be seen the set of histories of the pitch angle for chosen weighting matrices defined by eqs (25), (27), (29) and (31).

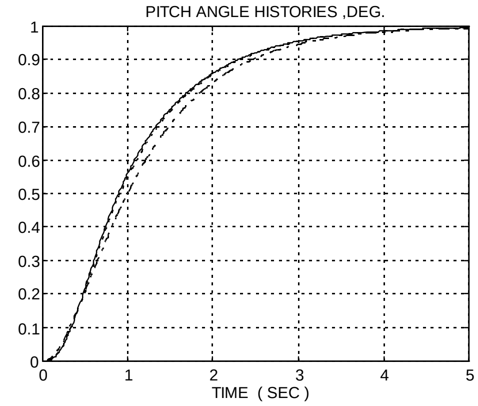


Figure 2. Pitch Attitude Control System Transient Response – Pitch Angle Histories
$\mathbf{Q}_1$ –dotted ; $\mathbf{Q}_2$ – dash-dot ; $\mathbf{Q}_3$ – solid

The inner loop transient behaviour can be seen in Figure 3. The solid line represents the set of weighting matrices, which is acceptable for the closed loop system dynamics.

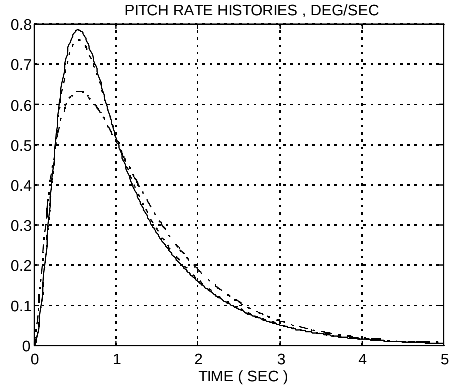


Figure 3. Pitch Attitude Control System Transient Response – Pitch Rate Histories

$Q_1$ – dotted ; $Q_2$ – dash-dot ; $Q_3$ – solid

From Figure 3. it is easily can be seen that any increase of weigths results in the increase of the maximum value of the pitch rate, which is often limited by the automatic flight control system.

The Bode diagram of the open control system can be seen in Figure 4. The LQG controller provides good disturbance rejection and noise suppressing ability for the control system. From Figure 4. it can be seen that the high frequency signals are well-damped by the control system. Use of the LQG design methodology for controller synthesis provides for the control system dynamic performances such as damping ratio, rise time, gain and phase margins in the pre-defined domain.

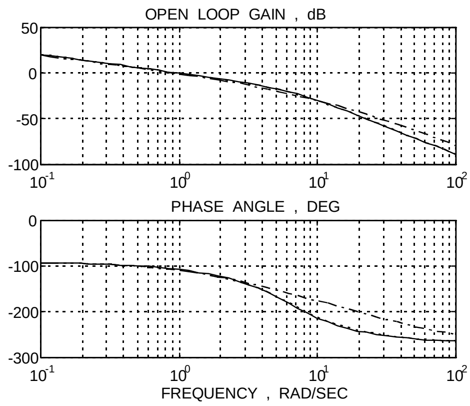


Figure 4. Bode Diagram of the Open Loop Pitch Attitude Control System

$Q_1$ – dotted ; $Q_2$ – dash-dot ; $Q_3$ – solid

The robustness of the closed loop control system can be analysed using Figure 5, which is the so-called closed loop complementary sensitivity transfer function.

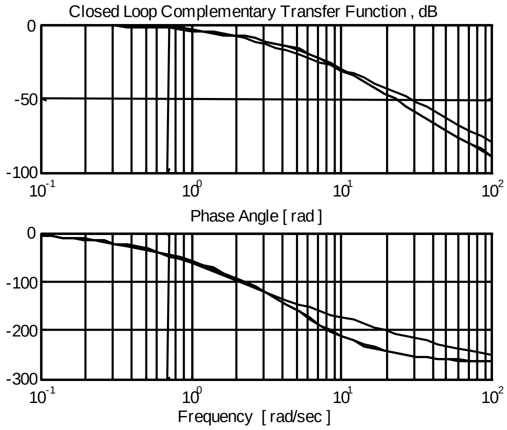


Figure 5. The Complementary Sensitivity of the Closed Loop Control System
$Q_1$ – dotted ; $Q_2$ – dash-dot ; $Q_3$ – solid

From Figure 5. it can be deduced that the strong robustness of the control system has been reached because of the strong damping of the high frequency signals e.g. sensor noise. The control system, which has been analyzed in this task is equipped with two and three degree-of-freedom electromechanical gyroscopes with the unmodeled natural frequencies (70-90) rad/sec. The designed controller allows to reduce the undesirable effects of the external and internal disturbances.

## CONCLUSIONS

This paper demonstrates the design of the LQG controller for the pitch attitude control system of the hypothetical fighter aircraft. The optimal design was carried out in the context of the determination of the LQG controller for a fighter, which is cruising at constant height of the flight. The motion of the aircraft was corrupted by external disturbances.

For solution of the synthesis of the dynamic LQG controller a special MATLAB® m-file was created by the author. The first set of the elements of the weighting matrices has been achieved using the inverse square rule. The acceptable dynamic performances have been reached applying the heuristically set weighting matrices.

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