SOLUTION OF THE CONTROL PROBLEMS USING MATLAB®

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The purpose of the author is to present the set of m-files created by himself for the solution of the control system analysis and design problems. The paper deals with the basic problems of the control system analysis and design, namely the control system building, control system model conversions, control system model properties, control system time domain behaviour, control system frequency domain behaviour and finally application of the gain selection for the control system are outlined in the present article. Each chapter contains examples presenting how to apply MATLAB® for solution of the analysis or the design tasks. The m-files attached to this paper had been created for use with MATLAB® 5.2 supplemented with Control System Toolbox.

CONTROL SYSTEM MODEL BUILDING WITH MATLAB®.

This section is based upon [10] concerning model building built-in functions of the Control System Toolbox. The big set of functions outlined in [10] can be given as follows:
append.m - append system dynamics, augstate.m - augment states as outputs, parallel.m - parallel system connection, series.m - series system connection, feedback.m - feedback system connection, cloop.m - close loops of system, blkbuild.m - build state-space system from block diagram, connect.m - block diagram modelling, ssselect.m - select subsystem from larger system, ssdelete.m - delete inputs, outputs or states from model, reg.m - form continuous controller/estimator from gain matrices, dreg.m - form discrete controller/estimator from gain matrices, estim.m - form continuous state estimator from gain matrix, destim.m - form discrete state estimator from gain matrix, conv.m - convolution of two polinomials, rmodel.m - generate random

continuous model, drmodel.m - generate random discrete model, ord2.m - generate A,B,C,D for a second-order system, pade.m - Padé approximation to time delay.

AN EXAMPLE FOR CONTROL SYSTEM MODEL BUILDING.

Let us consider the control system, which is shown in Figure 1.

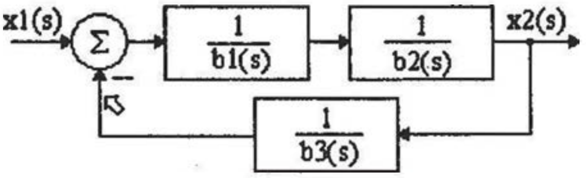

Figure 1.
Block Diagram of the Control System

Supposing that the control system feedforward path represents a series connection of two second order terms and the feedback path is a first order lag. Let us consider a second order terms with the general transfer funcion as defined below

$$Y_1(s) = \frac{1}{b_1(s)} = \frac{1}{s^2 + 2\xi_1\omega_1 s + \omega_1^2}, \quad Y_2(s) = \frac{1}{b_2(s)} = \frac{1}{s^2 + 2\xi_2\omega_2 s + \omega_2^2} \tag{1}$$

where ω is the natural frequency, ξ is the damping ratio. This mathematical model can be generated using "ord2.m" program of the Control System Toolbox.

Let the parameters of the second order terms be as they are given below:

$$\xi_1 = 0.8;\ \omega_1 = 2\ \text{rad/sec};\ \xi_2 = 0.7;\ \omega_2 = 3\ \text{rad/sec} \tag{2}$$

The first order lag in the feedback path has been considered with mathematical model as it shown below:

$$Y_3(s) = \frac{1}{b_3(s)} = \frac{1}{1 + 0.1s} \tag{3}$$

For the generating of the transfer function of the second order lag one can use the ord2.m program of the Control System Toolbox. For getting these transfer functions one should run the MATLAB® program N°1 (see APPENDIX A1.) under the prompt in the MATLAB® main window. One can get the following transfer functions:

$$Y_1(s) = \frac{1}{s^2 + 3{,}2s + 4};\ Y_2(s) = \frac{1}{s^2 + 4{,}2 + 9} \tag{4}$$

The transfer function of the feedforward path can be derived as

$$Y_4(s) = Y_1(s)Y_2(s) = \frac{1}{s^4 + 7{,}4s^3 + 26{,}44s^2 + 45{,}6s + 36} \tag{5}$$

Opening the closed loop in the feedback path at the arrow sign one can have the open loop transfer function as it given below:

$$Y_o(s) = Y_1(s)Y_2(s)Y_3(s) = \frac{1}{0{,}1s^5 + 1{,}74s^4 + 10{,}04s^3 + 31s^2 + 49{,}2s + 36} \tag{6}$$

The closed loop transfer function can be derived as follows

$$W_c(s) = \frac{x_2(s)}{x_1(s)} = \frac{Y_4(s)}{1 + Y_o(s)} = \frac{0{,}1s + 1}{0{,}1s^5 + 1{,}74s^4 + 10{,}04s^3 + 31s^2 + 49{,}2s + 37} \tag{7}$$

In this example the special set of m-files chosen by the author has been used for the application. For more details the interested in application of other programs of MATLAB® reader should refer to references [3,4,5,6,10].

CONTROL SYSTEM MODEL CONVERSIONS WITH MATLAB®.

This section is based upon reference [10] concerning model conversion built-in functions of the Control System Toolbox. The set of functions outlined in reference [10] can be given as follows:
c2d.m - continuous to discrete-time conversion, c2dm.m - continuous to discrete-time conversion with method, c2dt.m - continuous to discrete conversion with delay, d2c.m - discrete to continuous-time conversion, d2cm.m - discrete to

continuous-time conversion with method, residue.m - partial fraction expansion, poly.m - roots to polinomial conversion, ss2tf.m - state-space to transfer function conversion, ss2zp.m - state-space to zero-pole conversion, tf2ss.m - transfer function to state-space conversion, tf2zp.m - transfer function to zero-pole conversion, zp2tf.m - zero-pole to transfer conversion, zp2ss.m - zero-pole to state-space conversion.

AN EXAMPLE FOR CONTROL SYSTEM MODEL CONVERSIONS.

Let us consider the control system defined in the previous section in Fig.1. For the control system model conversion in this paper there are applied the "tf2ss.m", "tf2zp.m" and "c2d.m" programs of the Control System Toolbox of MATLAB®. The MATLAB® program for application of the mentioned above model conversions is given in Appendix A2. For getting the model conversions one should run the program in the MATLAB® main window.

Let us consider the closed loop transfer function defined by eq (7). The closed loop transfer function can be converted into the state-space model as follows

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu};\ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{8}$$

One can have the following matrices of the state-space model:

$$\mathbf{A} = \begin{bmatrix} -17,4 & -100,4 & -310 & -492 & -370 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix};\ \mathbf{C}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 10 \end{bmatrix};\ \mathbf{D} = 0 \tag{9}$$

The "tf2zp.m" program finds the SIMO control system factored transfer function form as it defined in [10] to be as follows:

$$W_c(s) = \frac{Z(s)}{p(s)} = \frac{[s-z_1][s-z_2]\ldots[s-z_m]}{[s-p_1][s-p_2]\ldots[s-p_n]} \tag{10}$$

where z_i are the zeros and p_j are the poles of the system.

After running the program given in Appendix A2. one can have the following zeros and poles of the control system:

$$\begin{aligned} & z_1 = -10;\ p_1 = -10{,}0103;\ p_{2,3} = -2{,}1666 \pm 2{,}1080\ i \\ & p_{3,4} = -1{,}5282 \pm 1{,}3075\ i \end{aligned} \tag{11}$$

If there is a need of conversion of the continuous control system into the discrete one you should use the "c2d.m" program. The dynamic system state equation defined by eq (8) can be converted into the following discrete state equation:

$$\mathbf{x}(n+1) = \mathbf{A}_d \mathbf{x}(n) + \mathbf{B}_d \mathbf{u}(n) \tag{12}$$

Before use this program programmer should define the sample time T_s. In this particular case sample time T_s has been supposed to be 0,01 sec. One can have the following matrices of eq (12):

$$\mathbf{A}_d = \begin{bmatrix} 0{,}8358 & -0{,}9347 & -2{,}8638 & -4{,}5256 & -3{,}3903 \\ 0{,}0092 & 0{,}9952 & -0{,}0147 & -0{,}0233 & -0{,}0175 \\ 0 & 0{,}01 & 1 & -0{,}0001 & -0{,}0001 \\ 0 & 0 & 0{,}01 & 1 & 0 \\ 0 & 0 & 0 & 0{,}01 & 1 \end{bmatrix} \tag{13}$$

$$\mathbf{B}_d = \begin{bmatrix} 0{,}0092 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

For having the discrete system time domain behaviour let the discrete output equation matrices be as follows:

$$\mathbf{C}_d = \begin{bmatrix} 0 & 0 & 0 & 1 & 10 \end{bmatrix};\ \mathbf{D}_d = 0 \tag{14}$$

Running the MATLAB® program in main window of MATLAB® gives the time domain behaviour of the discrete time system. The continuous system is firstly converted into the discrete system model. Secondly, the discrete state space model is converted to the transfer function model. For getting the discrete system time domain behaviour there has been used the "dstep.m" function of the

Control System Toolbox. Result of the computer simulation can be seen in Fig. 2.

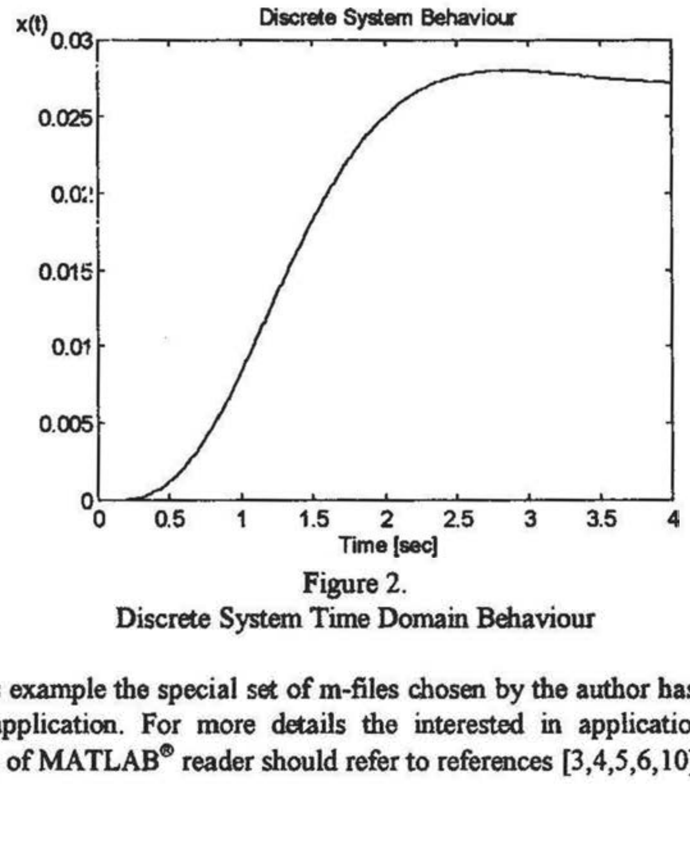

Figure 2.
Discrete System Time Domain Behaviour

In this example the special set of m-files chosen by the author has been used for the application. For more details the interested in application of other programs of MATLAB® reader should refer to references [3,4,5,6,10].

ANALYSIS OF THE CONTROL SYSTEM MODEL PROPERTIES WITH MATLAB®.

This section is based upon reference [10] concerning control system model properties built-in functions of the Control System Toolbox. The set of functions outlined in [10] can be given as follows: tzero.m - transmission zeros, eig.m - system eigenvalues, roots.m - poots of polinomial, ctrb.m - controllability matrix, obsv.m - observability matrix, damp.m - damping factors and natural frequencies, ddamp.m - discrete damping factors and natural frequencies, dcgain.m - continuous steady-state (D.C.) gain, ddgin.m - discrete steady-state (D.C.) gain, covar.m - continuous covariance response to white noise, dcovar. m - discrete covariance response to white noise, gram.m - controllability and observability gramians, dgram.m - discrete controllability and observability gramians, dsort.m - sort discrete eigenvalues by magnitude, esort.m - sort continuous eigenvalues by real part, printsys.m - special formatted print of system.

CONTROLLABILITY OF THE CONTROL SYSTEM.

Let us consider the linear time invariant (LTI) model of the control system given with its sate space model [1,2,7,8]:

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu}, \ \mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{15}$$

where **x** represents the n-dimensional state vector, **u** represents the m-dimensional input vector, **y** is the output vector, **A** is an (n×n) state matrix, **B** is an (n×m) input matrix, **C** is an (p×n) output matrix and **D** is an (p×m) direct feedforward matrix.

Controllability of the control system is a property of the coupling between the input and the state. Thus controllability involves matrices **A** and **B** of the state and output equations.

Definition 1. The control system is said to be *controllable* if and only if all initial state variables of the system, namely $x_i(0)$, can be transferred to their final state, say $x_i(T)$, in finite time by the application of the control vector **u**(t) [7]. If this condition takes places for all initial times t_0 and all initial states $x(t_0)$, the control system can be said *completely controllable*.

The LTI control system defined by eq (15) can be said completely controllable if and only if the (n×nm) controllability hypermatrix **M** has rank n, i. e. if matrix **M** has n linearly independent columns. The controllability matrix **M** can be defined as follows [7]:

$$\mathbf{M} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} & . & . & . & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} \tag{16}$$

For the single input systems the controllability matrix **M** becomes to the (n×n) square matrix and in this particular case the system can be noted as controllable if and only if **M** is non-singular, i. e. its determinant is non-zero one. The complete controllability is the sufficient condition for the closed loop stability.

Examples for the Controllability Test of the Control System.

Let us consider the lateral dynamics of the hypothetical aircraft given in [7] to be:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}}=\begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\beta}\\ \dot{\gamma}\end{bmatrix}=\begin{bmatrix}-10 & 0 & -10 & 0\\ 0 & -0,7 & 9 & 0\\ 0 & -1 & -0,7 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \beta\\ \gamma\end{bmatrix}+\begin{bmatrix}20 & 2,8\\ 0 & -3,13\\ 0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_a\\ \delta_r\end{bmatrix} \tag{17}$$

where ω_x is the roll rate, ω_y is the yaw rate, β is the sideslip angle, γ is the roll angle, δ_a is the deflection of the ailerons and finally δ_r is the rudder angular deflection, respectively.

Let us check the controllability of the aircraft for the particular case when there is supposed a malfunction in rudder deflection and the aircraft is controlled by deflection of the ailerons only. In this case one can consider only the first column of the input matrix **B** for checking of the controllability of the aircraft [3,4,5,6,9,10,11]. The MATLAB® program can be found in Appendix A3.

Firstly let us consider a case when the input is the aileron. The input matrix **B** becomes to the column vector as it shown below:

$$\mathbf{b}_1=\begin{bmatrix}20 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}} \tag{18}$$

Let us run the program under MATLAB® prompt outlined in Appendix A3. and get the controllability matrix $\mathbf{M}_1$. One can have:

$$\mathbf{M}_1=\begin{bmatrix}20 & -200 & 2000 & -20000\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 20 & 20 & 2000\end{bmatrix}, \tag{19}$$

which has rank 2, i.e. the aircraft is not controllable. The aircraft position can be controlled around longitudinal axis using only ailerons but it cannot be controlled around its vertical axis.

Let us consider the special case when the aircraft is controlled using the rudder. The input matrix **B** becomes as follows

$$\mathbf{b}_2=\begin{bmatrix}2,8 & -3,13 & 0 & 0\end{bmatrix}^{\mathrm{T}} \tag{20}$$

Finding the controllability matrix one can write that

$$\mathbf{M}_2 = \begin{bmatrix} -2{,}8 & -28 & 248{,}7 & -2443{,}2 \\ -3{,}1 & 2{,}2 & 26{,}6 & -58{,}1 \\ 0 & 3{,}1 & -4{,}4 & -23{,}6 \\ 0 & 2{,}8 & -28 & 248{,}7 \end{bmatrix}, \tag{21}$$

which has rank 4, i. e. the aircraft is controllable.

In McLean's textbook [1] there is an optimal control law synthesis example. Check the controllability of the aircraft dynamic model, which has been used for the controller design. The twin-engined jet fighter aircraft longitudinal motion dynamic model - for the flight conditions described in [1] - was as follows

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} v_x \\ \alpha \\ \omega_z \\ \vartheta \end{bmatrix} = \begin{bmatrix} -0{,}007 & 0.012 & 0 & -9{,}81 \\ -0{,}128 & -0{,}54 & 1 & 0 \\ 0{,}064 & 0{,}96 & -0{,}99 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ \alpha \\ \omega_z \\ \vartheta \end{bmatrix} + \begin{bmatrix} 0 \\ -0{,}036 \\ -12{,}61 \\ 0 \end{bmatrix} \delta_e, \tag{22}$$

where v_x is the horizontal speed, α is the angle-of-attack, ω_z is the pitch rate, ϑ is the pitch attitude and δ_e is the angular deflection of the elevator, respectively.

For the uncontrolled aircraft analysis let us run the MATLAB® program N⁰3 outlined in Appendix A3. and get the controllability matrix $\mathbf{M}_4$. One can have the controllability matrix as follows:

$$\mathbf{M}_3 = \begin{bmatrix} 0 & -0{,}0004 & 123{,}553 & -122{,}7619 \\ -0{,}036 & -12{,}5906 & 19{,}2483 & -50{,}6207 \\ -12{,}61 & 12{,}4493 & -24{,}4118 & 50{,}5535 \\ 0 & -12{,}6100 & 12{,}4493 & -24{,}4118 \end{bmatrix}, \tag{23}$$

which has rank of 4, i.e. all state variables of the aircraft are controllable ones.

OBSERVABILITY OF THE CONTROL SYSTEM.

Observability of the control system is the property of the coupling between the state and the output of the system. Thus, observability involves matrices **A** and **C** of the state and output equations.

Definition 2. The control system is said to be *observable* at t_0 if initial state vector $\mathbf{x}(t_0)$ can be determined from the output function $\mathbf{y}(t_0,T)$ (or output sequence) for $t_0 \le t \le T \le \infty$, where T is the finite time. If this condition takes place for all t_0 and $\mathbf{x}(t_0)$, the control system is said to be *completely observable*.

The LTI control system defined by eq (15) can be said completely observable if and only if the (n×np) observability hypermatrix **N** has rank n, i. e. if matrix **N** has n linearly independent columns. The observability matrix **N** can be defined as follows [1,7]:

$$\mathbf{N}=\left[\mathbf{C}^T \quad \mathbf{A}^T\mathbf{C}^T \quad \left(\mathbf{A}^T\right)^2\mathbf{C}^T \quad \left(\mathbf{A}^T\right)^3\mathbf{C}^T \quad . \; . \; . \quad \left(\mathbf{A}^T\right)^{n-1}\mathbf{C}^T\right] \tag{24}$$

If there is considered the single input system the observability matrix **N** is the square one of order (n×n) and the condition for control system observability requires that **N** be non-singular, i.e. its determinant is non-zero one with no pole-zero cancellations [1,7].

Examples for Observability Test of the Control System.

Let us consider the aircraft lateral motion dynamics defined by eq (17). Let us find the observability of the aircraft when the output of the aircraft is the roll rate ω_x. In this case the output matrix can be derived as follows [7]:

$$\mathbf{C}_1=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} \tag{25}$$

Using matrices **A** and **C** the MATLAB® program can be created for finding the observability matrix **N** [3,4,5,6,9,10,11]. For this purpose run the MATLAB® program N° 4 outlined in Appendix A4. One can have the observability matrix $\mathbf{N}_1$ as follows:

$$\mathbf{N}_1=\begin{bmatrix}1 & 0 & 0 & 0\\ -10 & 0 & -10 & 0\\ 100 & 10 & 107 & 0\\ -1000 & -114 & 984{,}9 & 0\end{bmatrix}, \tag{26}$$

which has rank of 3, thus the control system is unobservable.

Let us suppose that the output variable of the aircraft is the roll angle γ. The output matrix can be written as

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

One can have the observability matrix N_2 by running the MATLAB® program N°4 as follows:

$$N_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \end{bmatrix}, \tag{28}$$

which has the rank of 4, thus the system is completely observable.

Let us suppose that the output variables of the aircraft are the roll rate ω_x and the roll angle γ. The output matrix can be written as follows

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

One can have the observability matrix N_3 by running the MATLAB® program N°4. The rank of the observability matrix is 4, i.e. the system is completely observable.

ANALYSIS OF THE CONTROL SYSTEM TIME DOMAIN RESPONSE WITH MATLAB®.

Analysis of the time domain behaviour of the control systems is aided by the following functions of the Control System Toolbox of MATLAB® computer package: step.m - step response of the control system, dstep.m - discrete step response of the control system, initial.m - continuous system initial condition response, dinitial.m - discrete system unit sample response, lsim.m - continuous system simulation to arbitrary inputs, filter.m - SISO system z-transform simulation, ltitr.m - low level time response function.

An Example for the Analysis of the Control System Time Domain Behaviour.

Let us consider the system dynamics to be defined with the transfer function as given here:

$$Y(s) = \frac{1}{s^2 + 3s + 5} \qquad (30)$$

Find the step response of the system to a unit input step function. For this purpose let us run the MATLAB® program outlined in Appendix A5. After running of this program you can get the step response of the system to the sudden unit change in the input signal. Result of the computer simulation can be seen in Figure 3.

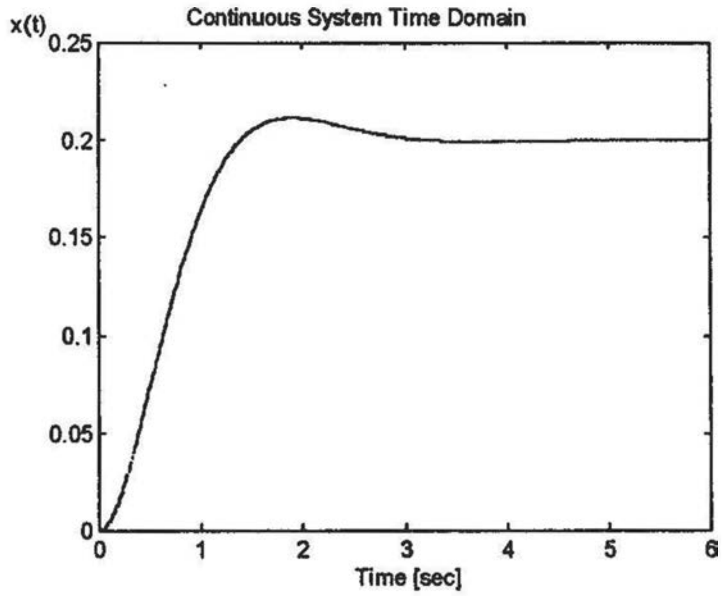

Figure 3.
Step Response of the Control System

From Figure 3. it can be deduced that the steady-state value of the output signal is 0,2. The second order term defined by transfer function (30) has overshoot related to the steady-state value of the ouput signal.

ANALYSIS OF THE CONTROL SYSTEM FREQUENCY DOMAIN BEHAVIOUR WITH MATLAB®.

Analysis of the frequency domain behaviour of the control systems is aided by the following functions of the Control System Toolbox of MATLAB®

computer package: bode.m - Bode plots, dbode.m - discrete Bode plots, fbode.m - fast Bode plots of continuous systems, margin.m - phase and gain margins, nichols.m - Nichols plots, dnichols.m - discrete Nichols plots, ngrid.m - grid lines for Nichols plot, nyquist.m - Nyquist plots, dnyquist.m - discrete Nyquist plots, sigma.m - continuous singular value frequency plots, dsigma.m - discrete singular value frequency plots, freqz.m - z-transform frequency response, freqs.m - Laplace-transform frequency response, ltifr.m - low level frequency response function.

An Example for the Analysis of the Control System Frequency Domain Behaviour.

Let us consider the system dynamics to be defined with the transfer function as follows:

$$Y(s)=\frac{1}{s^2+3s+5} \tag{31}$$

For getting the Bode plot of the system the "bode.m" program of the Control System Toolbox has been used. The gain and phase curves can be plotted running the program given in Appendix A6. The Bode plot can be seen in Fig. 4.

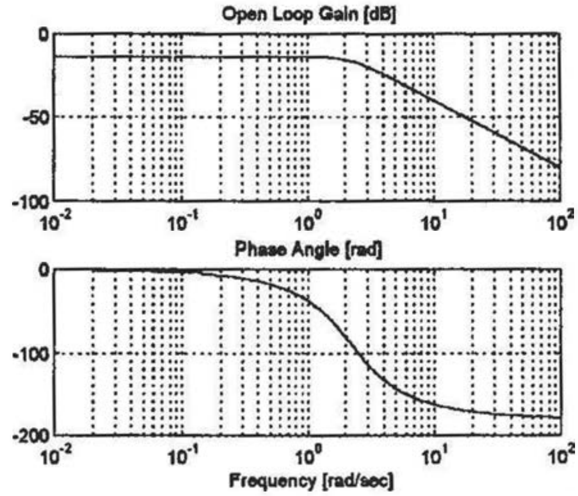

Figure 4.
Control System Bode Plot

From Figure 4. it can be seen that the system transfer function defined by eq (30) is the low pass filter and at frequencies bigger 2 rad/sec the system is damping the input signals. The phase angle goes to -180 degrees while frequency goes to infinity. For more information about plotting Bode plots the interested reader should refer to references [9, 10].

GAIN SELECTION FOR THE CONTROL SYSTEM WITH MATLAB®.

Analysis of the time domain behaviour of the control systems is aided by the following functions of the Control System Toolbox of MATLAB® computer package: lqr.m - linear-quadratic regulator design, lqr2.m - linear-quadratic regulator design using Schur method, lqry.m - regulator design with weighting on the outputs, dlqr.m - discrete linear-quadratic regulator design, dlqry.m - discrete regulator design with weighting on the outputs, lqe.m - linear-quadratic estimator design, lqe2.m - linear-quadratic estimator design using Schur method, lqew.m - general linear-quadratic estimator design, dlqe.m - discrete linear-quadratic estimator design, dlqew.m - general discrete linear-quadratic estimator design, lqrd.m - discrete regulator design from continuous cost function, lqed.m - discrete estimator design from continuous cost function, acker.m - SISO system pole placement, place.m - pole placement.

THE LINEAR QUADRATIC REGULATOR PROBLEM APPLIED FOR THE GAIN SELECTION IN THE AUTOMATIC FLIGHT CONTROL SYSTEMS

Meaning of the Modern Optimal Control Theory

The optimal control system is the special kind of control system, which can be characterized with high level dynamic performances. The optimal automatic flight control system can provide for the aircraft better flying and handling qualities. For judging when the control system is the optimal one there can be used and can be evaluated the following integral criterion [1]:

$$J = \int_{t_0}^{T} L(x, u, t)\, dt \rightarrow Min \qquad (32)$$

The system is regarded as the optimal one if between the starting point of the optimization t_0 and the final moment T the integral performance index has the minimized value. In case of minimization of the performance index the integral criterion is known as the cost function or pay-off function.

In that case when the control system can be characterized with limitations of its state variables the integral performance index can be rewritten as:

$$J = \int_{t_0}^{T} \left(\mathbf{e}^2 + \lambda \, \mathbf{u}^2 \right) dt \rightarrow \text{Min} \tag{33}$$

where λ is the Lagrange multiplier, $\mathbf{e}$ represents the error vector and can be derived as the difference between the actual and the commanded value of the state vector

$$\mathbf{e} = \mathbf{x} - \mathbf{x}_c \tag{34}$$

If one places some constraints on each control input u_j of the control vector $\mathbf{u}$ the integral performance index can be defined as:

$$J = \int_{t_0}^{T} \left(\mathbf{e}^T \mathbf{Q} \mathbf{e} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right) dt \rightarrow \text{Min} \tag{35}$$

where $\mathbf{Q} \geq 0$ and $\mathbf{R} > 0$ weighting matrices of the state variables and control inputs respectively.

If there is the problem of optimal control of the aircraft to be solved when the aircraft is considered to be stabilized at some trimmed flight state the error vector $\mathbf{e}$ is identical to the state variables of the aircraft so the performance index can be determined as [1,2,5,7,8]:

$$J = \int_{t_0}^{T} \left(\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{u}^T \mathbf{R} \mathbf{u} \right) dt \rightarrow \text{Min} \tag{36}$$

The quadratic terms $\mathbf{x}^T\mathbf{Q}\mathbf{x}$ and $\mathbf{u}^T\mathbf{R}\mathbf{u}$ can be explained as follows:

$$\mathbf{x}^T \mathbf{Q} \mathbf{x} = \begin{bmatrix} x_1 & . & . & . & x_n \end{bmatrix} \begin{bmatrix} q_1 & 0 & . & 0 & 0 \\ 0 & q_2 & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & q_{n-1} & 0 \\ 0 & 0 & . & 0 & q_n \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \tag{37}$$

$$= \begin{bmatrix} x_1 & . & . & . & x_n \end{bmatrix} \begin{bmatrix} q_1 x_1 \\ . \\ . \\ . \\ q_n x_n \end{bmatrix} = \sum_{i=1}^{n} q_i x_i^2$$

$$\mathbf{u}^T\mathbf{R}\,\mathbf{u} = [u_1 \;\; . \;\; . \;\; . \;\; u_n] \begin{bmatrix} r_1 & 0 & . & 0 & 0 \\ 0 & r_2 & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & r_{n-1} & 0 \\ 0 & 0 & . & 0 & r_n \end{bmatrix} \begin{bmatrix} u_1 \\ . \\ . \\ . \\ u_n \end{bmatrix} =$$

$$= [u_1 \;\; . \;\; . \;\; . \;\; u_n] \begin{bmatrix} r_1 u_1 \\ . \\ . \\ . \\ r_n u_n \end{bmatrix} = \sum_{j=1}^{n} r_j u_j^2 \qquad (38)$$

The integral criterion (36) is not the only but the most common one, which is used in aeronautical sciences.

When the integral performance index (36) is minimized through derivation of the optimal control vector **u** for the contol system with dynamics defined below

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad (39)$$

the problem is often called as the Linear Quadratic Problem.

The most important requirement, which is prescribed for the automatic flight control system is the quick reference signal tracking ability and the simultaneous capability as much as possible to reject the unwanted effects from the internal and external disturbances. The steady - state error may be minimized by increasing of the controller gain, but any increase of its tends the control system to the stable working boundary. These two requirements conflict and the control system design may be achieved as a consequence of some compromise.

The problem is to be solved is the next: for a given aircraft with the linear time invariant model find the control vector **u**, which will minimize the performance criterion (36). This problem also called as the minimum energy control problem. The linear optimal control law, which is minimizing the performance index (36) defined by [3,4,5,6] as follows :

$$\mathbf{u}^0(t) = -\mathbf{K}\,\mathbf{x}(t) \qquad (40)$$

where **K** is the static feedback gain matrix.

Let is asummed, that the aircraft's dymanics given by its state space representation in the body - fixed coordinate system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{41}$$

where **y** is the output vector, **C** and the **D** are the output and the feedforward matrices, respectively.

The block diagram of the control system built by eqs (40) and (41) for the particular case of **D**=0 can be seen in Figure 5.

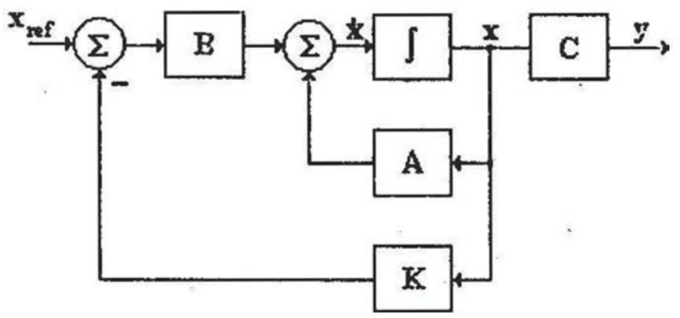

Figure 5.
Block Diagram of the Control System

Substituting the optimal control law (40) into the first equation of (41) results in

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} \tag{42}$$

Supposing that the eigenvalues of the matrix **A - BK** have negative values or, if there is any complex with negative real part. Substituting eq (40) into eq (36) result in the following equation:

$$J = \frac{1}{2}\int_0^\infty(\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{K}^T\mathbf{R}\mathbf{K}\mathbf{x})\,dt = \frac{1}{2}\int_0^\infty \mathbf{x}^T(\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K})\mathbf{x}\,dt \rightarrow \min \tag{43}$$

The minimization of the quadratic integral criterion (43) will be achieved using the second method of Liapounov. This method based upon assumption that for any **x** vector takes place the next equation:

$$\mathbf{x}^T(\mathbf{Q} + \mathbf{K}^T\mathbf{R}\mathbf{K})\mathbf{x} = -\frac{d}{dt}(\mathbf{x}^T\mathbf{P}\mathbf{x}) \tag{44}$$

where $\mathbf{P}$ is a positive definite or real symmetric matrix. Taking the derivative from the right side of eq (44) results in:

$$\mathbf{x}^T(\mathbf{Q}+\mathbf{K}^T\mathbf{R}\mathbf{K})\mathbf{x} = -\mathbf{x}^T\mathbf{P}\mathbf{x} - \mathbf{x}^T\mathbf{P}\mathbf{x} = -\mathbf{x}^T\left[(\mathbf{A}-\mathbf{B}\mathbf{K})^T\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{B}\mathbf{K})\right]\mathbf{x} \tag{45}$$

By the means of the second method of Liapounov for a given positive definite matrix $\mathbf{Q}+\mathbf{K}^T\mathbf{R}\mathbf{K}$ there is exists a positive definite matrix $\mathbf{P}$ such that

$$(\mathbf{A}-\mathbf{B}\mathbf{K})^T\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{B}\mathbf{K}) = -(\mathbf{Q}+\mathbf{K}^T\mathbf{R}\mathbf{K}) \tag{46}$$

The performance criterion J in this case can be evaluated as :

$$J = \frac{1}{2}\int_0^\infty \mathbf{x}^T(\mathbf{Q}+\mathbf{K}^T\mathbf{R}\mathbf{K})\mathbf{x}\,dt = -\left[\mathbf{x}^T\mathbf{P}\mathbf{x}\right]_0^\infty = -\mathbf{x}^T(\infty)\mathbf{P}\mathbf{x}(\infty)+\mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) \tag{47}$$

It was also assummed that all eigenvalues of the matrix $\mathbf{A}$ - $\mathbf{BK}$ has negative real parts. In this case one can write that $\mathbf{x}(\infty) \rightarrow 0$. Therefore, the quadratic integral criterion may be written as follows:

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) \tag{48}$$

The performance index J depends on the initial conditions $\mathbf{x}(0)$ and the matrix $\mathbf{P}$.

The weighting matrix $\mathbf{R}$ is the positive definite Hermitian or the real symmetric matrix and we can write that :

$$\mathbf{R} = \mathbf{T}^T\mathbf{T} \tag{49}$$

where is $\mathbf{T}$ is the nonsingular matrix. Then eq (46) can be rewritten in the following manner:

$$(\mathbf{A}^T-\mathbf{K}^T\mathbf{B}^T)\mathbf{P}+\mathbf{P}(\mathbf{A}-\mathbf{B}\mathbf{K})+\mathbf{Q}+\mathbf{K}^T\mathbf{T}^T\mathbf{T}\mathbf{K} = 0 \tag{50}$$

Doing multiplications in eq (50) it can be rewritten in the following manner:

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} + \left(-\mathbf{K}^T\mathbf{B}^T\mathbf{P} - \mathbf{PBK} + \mathbf{K}^T\mathbf{T}^T\mathbf{TK}\right) + \mathbf{Q} = 0 \tag{51}$$

It is known that $\mathbf{P} = \mathbf{P}^T$ and $\mathbf{R}^{-1} = \mathbf{T}^{-1}(\mathbf{T}^T)^{-1}$. The polinom in the brackets in eq (51) may be rewritten as it shown below

$$\mathbf{K}^T\mathbf{T}^T\mathbf{TK} - \mathbf{K}^T\mathbf{B}^T\mathbf{P} - \mathbf{PBK} = \mathbf{K}^T\mathbf{T}^T\mathbf{TK} - \mathbf{K}^T\left[\mathbf{T}^T\left(\mathbf{T}^T\right)^{-1}\right]\mathbf{B}^T\mathbf{P} - \mathbf{P}^T\mathbf{BK} +$$

$$+ \left(\mathbf{P}^T - \mathbf{P}\right)\mathbf{BR}^{-1}\mathbf{B}^T\mathbf{P} =$$

$$= \mathbf{K}^T\mathbf{T}^T\mathbf{TK} - \mathbf{K}^T\mathbf{T}^T\left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P} - \mathbf{P}^T\mathbf{B}\left(\mathbf{T}^{-1}\mathbf{T}\right)\mathbf{K} + \mathbf{P}^T\mathbf{B}\left[\mathbf{T}^{-1}\left(\mathbf{T}^T\right)^{-1}\right]\mathbf{B}^T\mathbf{P} -$$

$$- \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} =$$

$$= \left[\mathbf{K}^T\mathbf{T}^T - \mathbf{P}^T\mathbf{BT}^{-1}\right]\left[\mathbf{TK} - \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P}\right] - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} =$$

$$= \left[\mathbf{TK} - \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P}\right]^T\left[\mathbf{TK} - \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P}\right] - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} \tag{52}$$

Eq (51) can be rewritten with consideration of eq (52). One can write that

$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} + \left[\mathbf{TK} - \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P}\right]^T\left[\mathbf{TK} - \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P}\right] - \mathbf{PBR}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = 0 \tag{53}$$

The minimization of J requires the minimization of

$$\left[\mathbf{TK} - \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P}\right]^T\left[\mathbf{TK} - \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P}\right] \tag{54}$$

Since eq (54) is nonnegative, its minimum occurs when it is zero or, in that case when

$$\mathbf{TK} = \left(\mathbf{T}^T\right)^{-1}\mathbf{B}^T\mathbf{P} \tag{55}$$

Hence the optimal feedback gain matrix can be found as follows by [18,19]:

$$\mathbf{K}^{o} = \mathbf{T}^{-1}\left(\mathbf{T}^{T}\right)^{-1}\mathbf{B}^{T}\mathbf{P} = \mathbf{R}^{-1}\mathbf{B}^{T}\mathbf{P} \tag{56}$$

Equation (56) determining the feeback gain matrix of the optimal control law defined by eq (40). In case when eq (55) takes place eq (53) can be rewritten in the following manner:

$$\mathbf{A}^{T}\mathbf{P} + \mathbf{PA} - \mathbf{PBR}^{-1}\mathbf{B}^{T}\mathbf{P} + \mathbf{Q} = 0 \tag{57}$$

Equation (57) also called as the reduced - matrix Ricatti equation (algebraic Ricatti equation - ARE) for the time invariant cost matrix **P**.

The optimal control law synthesis contents the following two steps:
1, solution of the ARE - eq (57) - in order to get the matrix **P**,
2, substituting matrix **P** into eq (56). The resulting feedback gain matrix **K** is an optimal for the chosen **Q** and **R** matrices.

An Example for the Application of the LQR Problem.

Let us consider the directional control system of the generic aircraft. The block diagram of the control system can be seen in Figure 8.2. During solution of this problem there was considered the so called single degree of fredom approximation of the mathematical model of the aircraft [8,9,12].

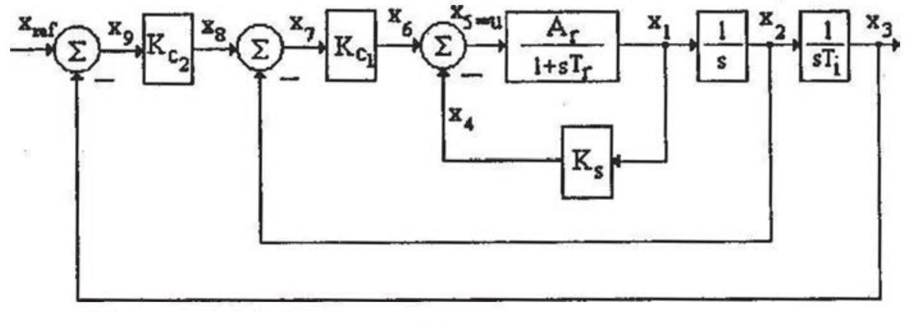

Figure 6.
Block Diagram of the Directional Control System

The state variables can be deduced from Figure 6. They are as follows:

$$x_1 = \omega_x \; ; \; x_2 = \gamma \; ; \; x_3 = \psi \tag{58}$$

where ω_x is the roll rate, γ is the roll angle and ψ is the direction angle.

The design problem can be formulated as follows: find the optimal control law $\mathbf{u}_0$ for the aircraft, which provides for the control system the prescibed dynamic characteristics. This problem can be solved using methodology outlined in this chapter. For this it is necessary to determine the state equation and the output equation of the control system. From Figure 6. one can deduce the following equations:

$$x_1 = u\,\frac{A_r}{1+sT_r} \rightarrow u = x_1\frac{1+sT_r}{A_r} = \frac{x_1}{A_r} + sx_1\frac{T_r}{A_r}$$
$$\text{or } \dot{x}_1 = -\frac{x_1}{T_r} + u\,\frac{A_r}{T_r} \quad \text{és} \quad x_4 = K_S x_1 \;;\quad u = x_6 - x_4 = x_6 - K_S x_1 \tag{59}$$

Secondly,

$$x_2 = \frac{x_1}{s} \rightarrow \dot{x}_2 = x_1 \;;\; x_6 = K_{C_1}x_7 \;;\; x_7 = x_8 - x_2 \;;\; x_8 = K_{C_2}x_9$$
$$u = -K_S x_1 + K_{C_1}x_7 = -K_S x_1 + K_{C_1}\left(x_8 - x_2\right) = -K_S x_1 - K_{C_1}x_2 + K_{C_1}x_8 = \tag{60}$$
$$= -K_S x_1 - K_{C_1}x_2 + K_{C_1}K_{C_2}x_9\,,$$

Thirdly,

$$x_3 = \frac{x_2}{sT_i} \rightarrow \dot{x}_3 = \frac{x_2}{T_i} \;;\; x_9 = x_{ref} - x_3 \tag{61}$$

Let us consider the zero value reference signal $x_{ref} = 0$, therefore $x_9 = -x_3$. The control vector can be determined as

$$\mathbf{u} = -K_S x_1 - K_{C_1}x_2 - K_{C_1}K_{C_2}x_3 = -\begin{bmatrix}K_S & K_{C_1} & K_{C_1}K_{C_2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = -\mathbf{K}\mathbf{x} \tag{62}$$

where $\mathbf{K} = \begin{bmatrix}K_S & K_{C_1} & K_{C_1}K_{C_2}\end{bmatrix}$ is the static feedback gain matrix and the state vector can be derived as $\mathbf{x} = \begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}^T$.

Thus one can write the state equations of the control system in the following manner:

$$\dot{x}_1 = -\frac{x_1}{T_r} + u\frac{A_r}{T_r} \;;\; \dot{x}_2 = x_1 \;;\; \dot{x}_3 = \frac{x_2}{T_i}, \tag{63}$$

or using the well-known matrix notation one can write that

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u},\ \mathbf{y} = \mathbf{C}\mathbf{x} \tag{64}$$

Matrices and vectors of the model are found to be followings

$$\mathbf{x}=\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}} \;;\; \mathbf{u}=u \;;\; \mathbf{B}=\begin{bmatrix} \frac{A_r}{T_r} & 0 & 0 \end{bmatrix}^{\mathrm{T}} \;;\; \mathbf{A}=\begin{bmatrix} -\frac{1}{T_r} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{T_i} & 0 \end{bmatrix} \;;\; \mathbf{C}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{65}$$

Let us the dynamics and kinematics of the aircraft be as follows:

$$\mathrm{A_r} = 2{,}86\ \mathrm{s}^{-1} \;;\; \mathrm{T_r} = 0{,}568\ \mathrm{s} \;;\; \mathrm{T_i} = 17\ \mathrm{s} \tag{66}$$

The state and the input matrices of the eq(8.33) are to be

$$\mathbf{A} = \begin{bmatrix} -1{,}7605 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0{,}0588 & 0 \end{bmatrix} \;;\; \mathbf{B} = \begin{bmatrix} 5{,}0352 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{67}$$

Let us analyze the uncontrolled aircraft time domain behaviour. During analysis the step response of the aircraft is determined for the angular deflection of the ailerons. The aircraft dynamics can be characterized with the dynamics of the first order system. The roll rate transient behaviour can be seen in Figure 7. From Figure 6. the integral relationship between roll rate and roll angle, and also between roll angle and the direction angle can be deduced. The roll angle transient behaviour and the direction angle transient behaviour can be seen in Figure 8. and Figure 9. For getting of results of the control system design let us run the MATLAB® program outlined in Appendix A6.

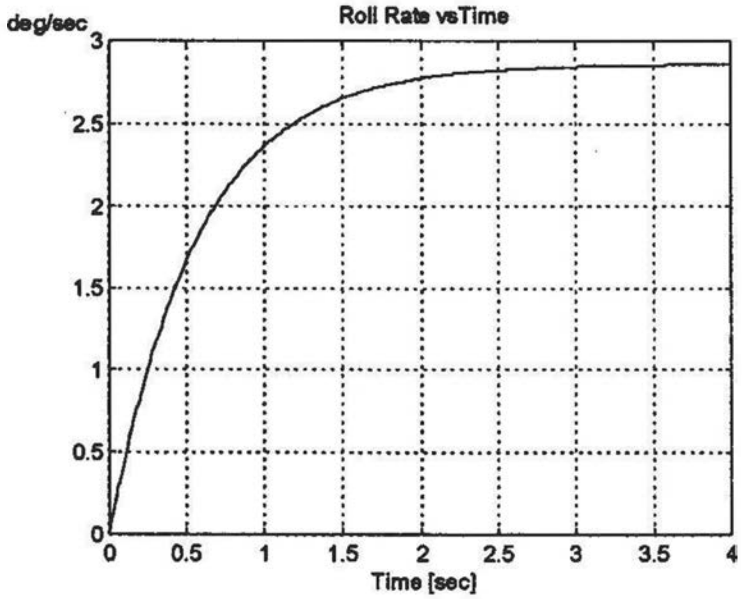

Figure 7.
Time Domain Behaviour of the Aircraft Roll Rate

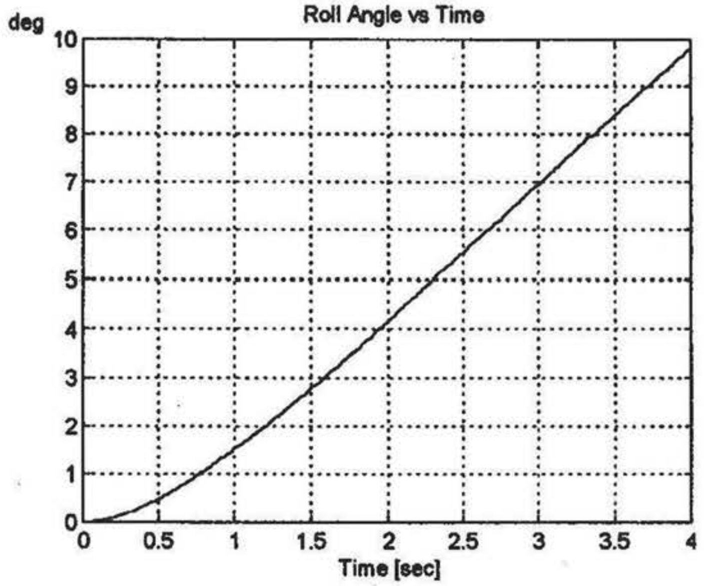

Figure 8.
Time Domain Behaviour of the Aircraft Roll Angle

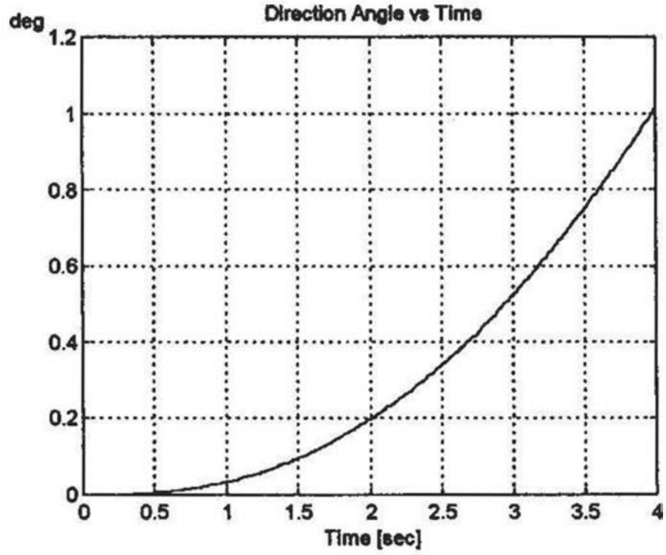

Figure 9.
Time Domain Behaviour of the Aircraft Direction Angle

Let us solve the LQR problem for the direction control system. The aim of the control law synthesis is to provide for the closed loop system the following eigenvalues [18,19,20,21]:

$$\lambda_{1,2} = -0{,}7 \pm 0{,}8i \;;\; \lambda_3 = -2{,}8 \tag{68}$$

The weighting matricen in integral performance index (36) can be determined using the inverse square rule. Most of flight control systems can be characterized with existence in the flight control systems several limitations. Limitations may be developed by designer in order to provide necessary flight safety characteristics or limitations can be achived by the flying characteristics of the given type of the aircraft.

During control law synthesis there are supposed the following limitations in the flight control system:

$$|\omega_{x\max}| = 18^0 / \sec,\ |\gamma_{\max}| = 90^0,\ |\psi_{\max}| = 20^0,\ |\delta_{a\max}| = 3^0 \tag{69}$$

Applying the inverse square rule for the selection of the elements of weighting matrices **Q** and **R** one can write that [10]:

$$\mathbf{Q}_1 = \begin{bmatrix} 3{,}0864 * 10^{-3} & 0 & 0 \\ 0 & 1{,}2345 * 10^{-4} & 0 \\ 0 & 0 & 0{,}0025 \end{bmatrix};\ \mathbf{R}_1 = [0{,}1111] \tag{70}$$

For the solution of the LQR problem the "lqr2.m" file of the Control System Toolbox of the computer package MATLAB® (Version 5.2) has been used [10,11].

Minimization of the cost function (36) using matrices (70) results in the following static feedback gain matrix:

$$\mathbf{K}_1 = \begin{bmatrix} 0{,}0808 & 0{,}0887 & 0{,}15 \end{bmatrix}, \tag{71}$$

or in the other manner

$$K_s = 0{,}0808,\ K_{c_1} = 0{,}0887,\ K_{c_2} = 1{,}6910 \tag{72}$$

Let us derive the closed loop eigenvalues. They are as follows [19,20,21,22,23]:

$$\lambda_{1,2} = -0{,}1087 \pm 0{,}1047i,\ \lambda_3 = -1{,}9499 \tag{73}$$

It is easily can be seen that the closed loop system with its dynamic characteristics is not satisfy the design requirements defined by eq (68). During time domain analysis of the directional control system a step change of the directional angle has been considered. Results of the computer simulation of the closed loop system can be seen in Figures 10., 11., and 12.

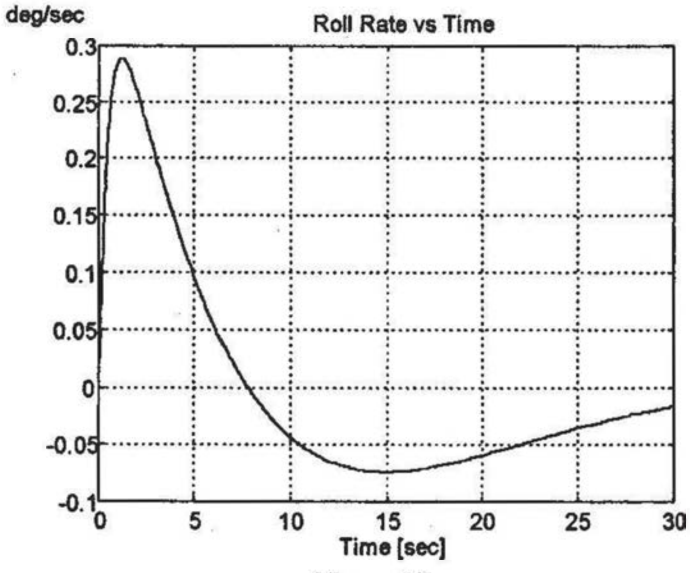

Figure 10.
Closed Loop System Behaviour - Time Domain Analysis of the Roll Rate

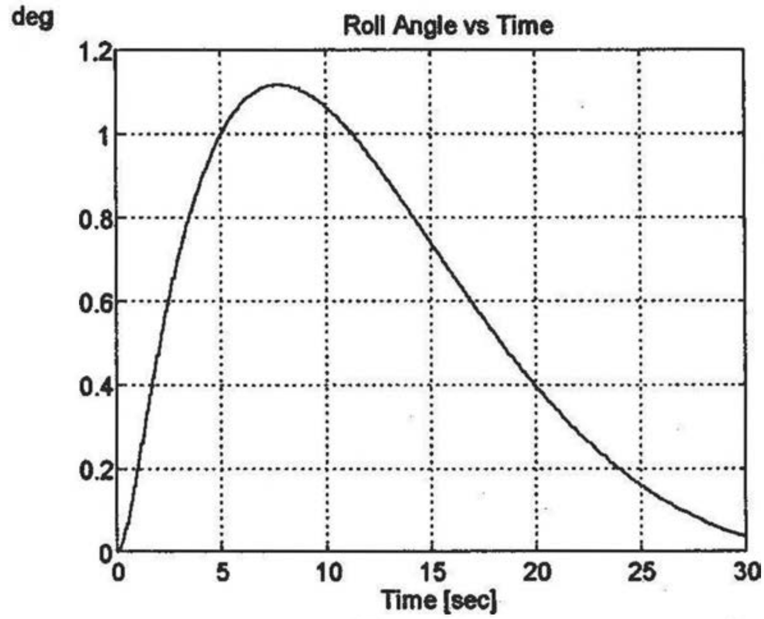

Figure 11.
Closed Loop System Behaviour - Time Domain Analysis of the Roll Angle

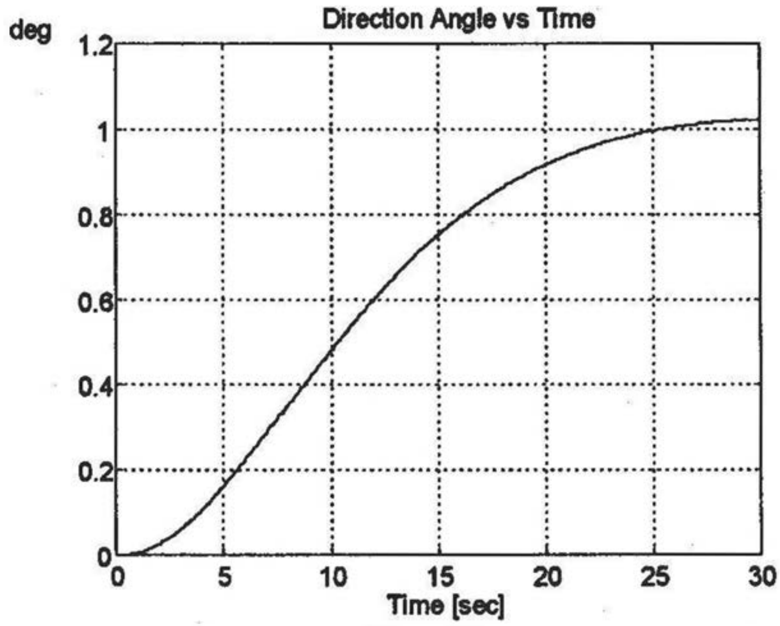

Figure 12.
Closed Loop System Behaviour - Time Domain Analysis of the Direction Angle

Let the parameters of the weighting matrices be as they listed below:

$$\mathbf{Q_2} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3{,}0864e-2 & 0 \\ 0 & 0 & 2000 \end{bmatrix}; \quad \mathbf{R_2} = [20] \tag{74}$$

In this case the static feedback gain matrix can be found minimizing eq (36). The LQR problem has been solved for the set of weighting matrices given with eq (74). The static feedback gain matrix has been found to be as [9,10]:

$$\mathbf{K_2} = \begin{bmatrix} 0{,}4633 & 0{,}9786 & 10{,}00 \end{bmatrix}, \tag{75}$$

or in the other manner

$$K_s = 0{,}4633,\ K_{c_1} = 0{,}9786,\ K_{c_2} = 10{,}2186 \tag{76}$$

The closed loop eigenvalues have been found to be as follows

$$\lambda_{1,2} = -0{,}7171 \pm 0{,}7740i,\ \lambda_3 = -2{,}6593 \tag{77}$$

It is easily can be seen that closed loop eigenvalues defined by eq (68), which represents the design specification are very close to the closed loop eigenvalues determined for the directional control system with the static feedback gain $\mathbf{K_2}$.

The results of the flight control system time domain analysis in case when there is analized the reference signal tracking ability can be seen in Figures 13., 14., and 15.

During analysis the step change in the reference direction angle has been considered. The results of the time domain analysis have been calculated using MATLAB® computer package supplemented with Control System Toolbox [9,10].

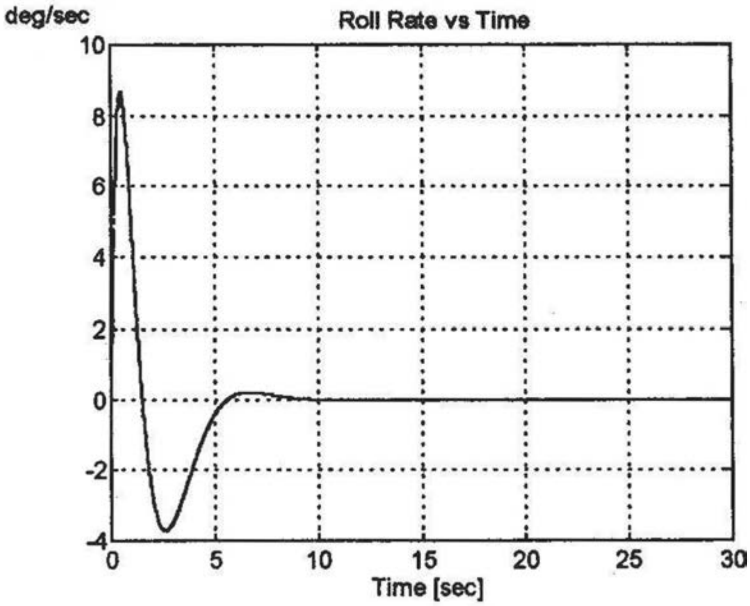

Figure 13.
Closed Loop System Behaviour - Time Domain Analysis of the Roll Rate

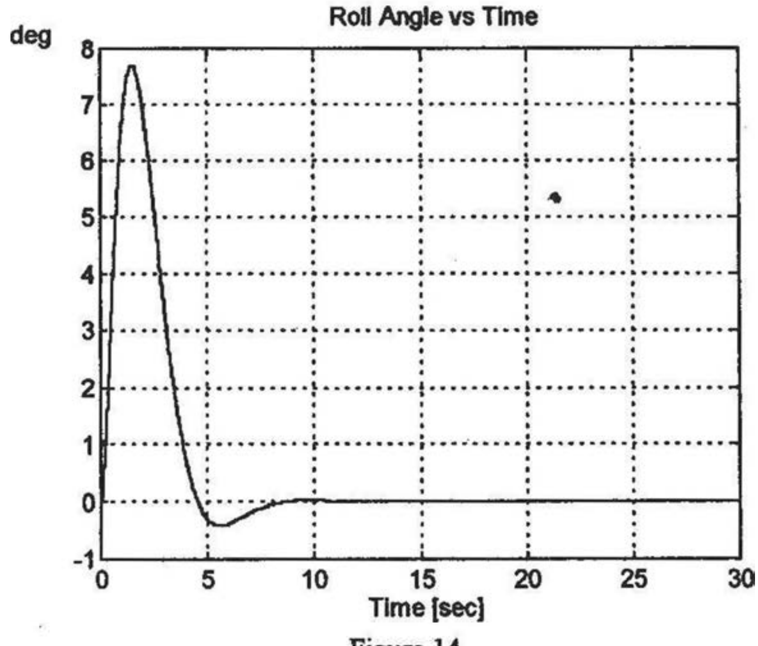

Figure 14.
Closed Loop System Behaviour - Time Domain Analysis of the Roll Angle

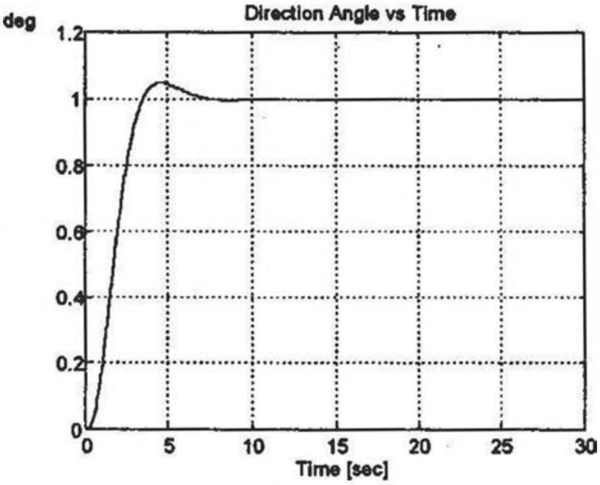

Figure 15.
Closed Loop System Behaviour - Direction Angle Time Domain Analysis

The closed loop damping ratios are as folows

$$\xi_{1,2} = 0{,}6796,\ \xi_3 = 1 \tag{78}$$

The transient response time is close to 7 seconds, which satisfies design specifications prescribed for the control system [8,12]. The comparision of two designed systems' time domain behaviour can be seen in Figure 16.

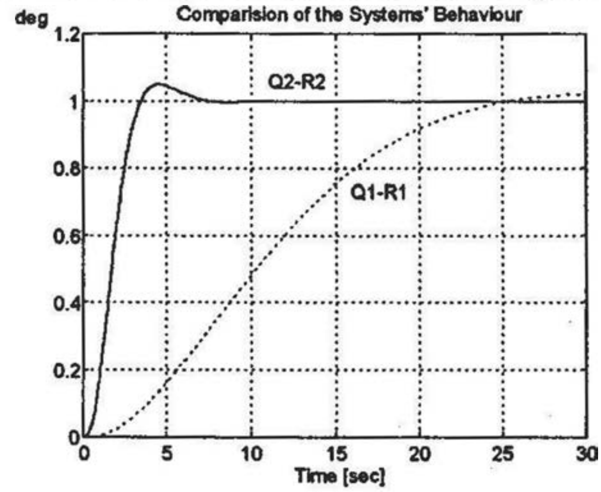

Figure 16.
Step Response Of the Directional Control Systems with Different Weights:
Q_1, R_1 - "...", Q_2, R_2 - "–"

From Figure 16. it is easily can be seen that changing parameters of the weighting matrices, as it was defined earlier, results in better dynamic performances. Time of the transient response has been decreased. The overshoot of the closed system and the damping ratio are also in that range of general design specifications.

CONCLUSIONS

In this paper some problems of control systems' analysis and design had been highlighted. A set of MATLAB® programs has been created by the author for presenting solutions of control problems using computer package MATLAB®. The MATLAB® computer program is widely used during flight control systems' onground design and testing.

ACKNOWLEDGEMENTS

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APPENDICES

A1. The MATLAB® Program N°1 for the Control System Model Building.

```
% Transfer functions: Y1(s)=a1(s)/b1(s) and Y2(s)=a2(s)/b2(s)
[a1,b1]=ord2(2,0.8)
pause
[a2,b2]=ord2(3,0.7)
pause

% Derivation of the feedforward path transfer function
% Transfer function of the series connection of two terms: Y3(s)=Y1(s)*Y2(s)
[a3,b3]=series(a1,b1,a2,b2)
pause

% Definition of the feedback path transfer function: Y4(s)=a4(s)/b4(s)
a4=[0 1];
b4=[0.1 1];
pause

% Derivation of the open loop transfer function after opening the loop  in the
% feedback path at the point, which is shown in Figure 1. with arrow.
% Transfer function of the terms located in the open loop:
Yo(s)=Y1(s)*Y2(s)*Y4(s)
[a5,b5]=series(a3,b3,a4,b4)
pause

% Derivation of the closed loop transfer function:
% Wc(s)=Y3(s)/[1+Y3(s)*Y4(s)]
[a6,b6]=feedback(a3,b3,a4,b4)
pause
```

A2. The MATLAB® Program N°2 for the Control System Model Conversions.

```
% The closed loop transfer function derived in Appendix A1 is as follows
a=[0 0 0 0 0.1 1];
b=[0.1 1.74 10.04 31 49.2 37];
```

```
% Control System Model Conversion to the State-Space
[A,B,C,D]=tf2ss(a,b)
pause

% Control System Model Conversion to the Zero-Pole Model
[z,p,k]=tf2zp(a,b)
pause

% Conversion of the Continuous Control System to Discrete-Time System
Ts=0.02;
[Ad,Bd]=c2d(A,B,Ts)
pause

% Definition of the Matrices of the Output Equation
Cd=[0 0 0 1 1];
Dd=0;

% Conversion of the Discrete State-Space Model to Transfer Function
[ad,bd]=ss2tf(Ad,Bd,Cd,Dd)
pause

% Analysis of the Time Domain Behaviour
dstep(ad,bd);
pause
```

A3. The MATLAB® Program N⁰3 for the LTI System Controllability Test.

```
% The first example.
% Definition of the state matrix of the uncontrolled aircraft
A = [-10 0 -10 0;0 -0.7 9 0;0 -1 -0.7 0;1 0 0 0];
% 1st case - input of the aircraft is the angular deflection of the ailerons.
% The input matrix is as follows:
b1 = [20;0;0;0];
% The check of the controllability of the aircraft means the derivation of the
% controllability matrix of the % uncontrolled aircraft
M1=ctrb(A,b1)
pause
% Determination of the uncontrollable states of the aircraft
uncoM1=length(A)-rank(M1)
pause
```

```
% 2nd case-input of the aircraft is the angular deflection of the ailerons.
% The input matrix is as follows:
b2 = [2.8;-3.13;0;0];
% The check of the controllability of the aircraft means the derivation of the
% controllability matrix of the uncontrolled aircraft.
M2=ctrb(A,b2)
pause

% Determination of the uncontrollable states of the aircraft.
uncoM2=length(A)-rank(M2)
pause

% 3rd case - inputs of the aircraft are both ailerons and rudder.
B = [20 2.8;0 -3.13;0 0;0 0];
% The check of the controllability of the aircraft means the derivation of the
% controllability matrix of the uncontrolled aircraft
M3=ctrb(A,B)
pause

% Determination of the uncontrollable states of the aircraft.
uncoM3=length(A)-rank(M3)
pause

% Second example - The twin-engined jet fighter aircraft controllability test.
A1 = [-0.007 0.012 0 -9.81;-0.128 -0.54 1 0;0.064 0.96 -0.99 0;0 0 1 0];
B1 = [0;-0.036;-12.61;0];
% The check of the controllability of the aircraft means the derivation of the
% controllability matrix of the uncontrolled aircraft
M4=ctrb(A1,B1)
pause

% Determination of the uncontrollable states of the aircraft.
uncoM4=length(A1)-rank(M4)
pause
```

A4. The MATLAB® Program N°4 for the LTI System Observability Test.

```
% The state matrix of the aircraft.
A = [-10 0 -10 0;0 -0.7 9 0;0 -1 -0.7 0;1 0 0 0];
% 1st case - output of the aircraft is the roll rate.
```

```
C1 = [1 0 0 0];
% Checking the observability of the aircraft.
N1=obsv(A,C1)
pause

% Determination of the unobservable states of the aircraft.
unobN1=length(A)-rank(N1)
pause

% 2nd case - ouput is roll angle of the aircraft.
C2 = [0 0 0 1];
pause

% Checking the observability of the aircraft.
N2=obsv(A,C2)
pause

% Determination of the unobservable states of the aircraft.
unobN2=length(A)-rank(N2)
pause

% 3rd case - ouputs of the aircraft are roll rate and roll angle.
C3 = [1 0 0 0;0 0 0 1];
pause

% Checking the observability of the aircraft.
N3=obsv(A,C3)
pause

% Determination of the unobservable states of the aircraft.
unobN3=length(A)-rank(N3)
pause
```

A5. The MATLAB® Program N°5 for the Analysis of the LTI System Time Domain Behaviour.

```
% APPENDIX A2 - AN EXAMPLE FOR CONTROL SYSTEM TIME
% DOMAIN BEHAVIOUR ANALYSIS
a=[0 0 0 0 0.1 1];
b=[0.1 1.74 10.04 31 49.2 37];
pause
```

```
% Definition of the time domain
t=[0:0.01:6];
pause

% Step Response of the Closed Loop System
y=step(a,b,t);

% Plotting the Curve x2(t)
plot(t,y,'-w')
xlabel('Time [sec]'), ylabel('x2(t)'), title('Closed Loop Behaviour')
pause
clg
```

A6. The MATLAB® Program N°6 for the Analysis of the LTI System Frequency Domain Behaviour.

```
% APPENDIX A6 - AN EXAMPLE FOR ANALYSIS OF THE CONTROL
% SYSTEM FREQUENCY  DOMAIN BEHAVIOUR.
% The system dynamics is given with the model of the second order lag:
a=[0 0 1];
b=[1 3 5];
pause

% Definition of the Frequency Domain Applied for Analysis
om=logspace(-2,2,1000);

% Derivation of the Open Loop Gain and Phase
[mag,phase]=bode(a,b,om);
mag=20*log10(mag);
phase=phase';

subplot(211),
semilogx(om,mag,'-w'),grid
title('Open Loop Gain [dB]')
subplot(212),
semilogx(om,phase,'-w'),grid
xlabel('Frequency [rad/sec]')
title('Phase Angle [rad]')
pause
```

A7. The MATLAB® Program N°7 for the Computer Aided Design of the Control Systems. Solution of the LQR Problem.

```
% Definition of the Aircraft Dynamic State-Space Model.
A = [-1.7605 0 0;1 0 0;0 0.0588 0];
B = [5.0352;0;0];
C = [1 0 0;0 1 0;0 0 1];
D = [0;0;0];

% Definition of the Time Domain for the Uncontrolled Aircraft Transient
% Response Analysis.
t = [0:0.01:4];

% Controllability Test of the Uncontrolled Aircraft.
co=ctrb(A,B),rank(co),
pause

% Observability Test of the Uncontrolled Aircraft.
ob=obsv(A,C),rank(ob),
pause

% Derivation of the Open Loop Eigenvalues.
format short
damp(A)
pause

% Analysis of the Transient Response of the Uncontrolled Aircraft.
y1=step(A,B,C,D,1,t);
omx=[1 0 0]*y1';gam=[0 1 0]*y1';pszi=[0 0 1]*y1';
plot(t,omx,'-w'),grid,
title('Roll Rate v Time'),
xlabel('Time [sec]'),ylabel('deg/sec')
pause
plot(t,gam,'-w'),grid,
title('Roll Angle v Time'),
xlabel('Time [sec]'),ylabel('deg')
pause
plot(t,pszi,'-w'),grid,title('Direction Angle v Time')
xlabel('Time [sec]'), ylabel('deg')
pause
```

```
% Solution of the LQR Problem Using Inverse Square Rule for Selection of the
% Weighting Matrices.
Q1 =[0.0030864 0 0;0 0.00012345 0;0 0 0.0025];
R1 =[0.1111];

% Finding the Optimal Feedback Static Gain.
K1 = lqr2(A,B,Q1,R1)
pause

% Unit Step Response of the Designed System. 1st Approache.
t1=[0:0.01:30];
AA1=A-B*K1; BB1=B*K1(3); CC1=eye(3); DD1=D;
y2=step(AA1,BB1,CC1,DD1,1,t1);
pause

omx1=[1 0 0]*y2';gam1=[0 1 0]*y2';pszi1=[0 0 1]*y2';
plot(t1,omx1,'-w'),grid,
title('Roll Rate v Time'),xlabel('Time [sec]'),ylabel('deg/sec')
pause
plot(t1,gam1,'-w'),grid,
title('Roll Angle v Time'),xlabel('Time [sec]'),ylabel('deg')
pause
plot(t1,pszi1,'-w'),grid,
title('Direction Angle v Time'),xlabel('Time [sec]'),ylabel('deg')
pause

% Derivation of the Closed Loop System Eigenvalues.
format short
damp(AA1)
pause

% Solution of the LQR Problem Using Inverse Square Rule and Additional
% Change of the Weighting Matrices Based on Heuristic Method.
Q2 =[3 0 0;0 0.030864 0;0 0 2000];
R2 =[20];
% Derivation of the Optimal Feedback Static Gain.
K2 = lqr2(A,B,Q2,R2)
pause
```

```
% Unit Step Response of the Designed System.
t2=[0:0.01:30];
AA2=A-B*K2; BB2=B*K2(3); CC2=eye(3); DD2=D;
y3=step(AA2,BB2,CC2,DD2,1,t2);
pause

omx2=[1 0 0]*y3';gam2=[0 1 0]*y3';pszi2=[0 0 1]*y3';
plot(t2,omx2,'-w'),grid
title('Roll Rate v Time'),xlabel('Time [sec]'),ylabel('deg/sec')
pause
plot(t2,gam2,'-w'),grid
title('Roll Angle v Time'),xlabel('Time [sec]'),ylabel('deg')
pause
plot(t2,pszi2,'-w'),grid
title('Direction Angle v Time'),xlabel('Time [sec]'),ylabel('deg')
pause

% Derivation of the Closed Loop Eigenvalues.
format short
damp(AA2)
pause

% Comparision of Two Designed System.

plot(t1,pszi1,':w',t2,pszi2,'-w'),grid
xlabel('Time [sec]'),ylabel('deg'),title('Comparision of the Systems Behaviour')
pause
gtext('Q1-R1')
pause
gtext('Q2-R2')
```